Université de Montréal

A class of bivariate Erlang distributions and ruin probabilities in multivariate risk models

par

Ionica Groparu-Cojocaru

Département de mathématiques et de statistique
Faculté des arts et des sciences

Thèse présentée à la Faculté des études supérieures
en vue de l'obtention du grade de
Philosophiæ Doctor (Ph.D.)
en Mathématiques

Orientation Mathématiques Appliquées

novembre 2012

Université de Montréal

Faculté des études supérieures

Cette thèse intitulée

A class of bivariate Erlang distributions and ruin probabilities in multivariate risk models

présentée par

Ionica Groparu-Cojocaru

a été évaluée par un jury composé des personnes suivantes :

Mylène Bédard
(président-rapporteur)

Louis Doray
(directeur de recherche)

Manuel Morales
(co-directeur)

José Garrido
(membre du jury)

Emiliano Valdez
(examinateur externe)

Mylène Bédard
(représentant du doyen de la FES)

Thèse acceptée le:
8 novembre 2012

SOMMAIRE

Cette thèse est consacrée à la modélisation de la dépendance avec des applications potentielles en science actuarielle et est divisée en deux parties: la première considère la dépendance dans le contexte de l'analyse des données de survie bivariées et la deuxième, liée à la théorie du risque, traite de la dépendance entre les classes d'affaires en assurance.

La première partie est présentée sous la forme d'un article de recherche au chapitre 3 de la thèse. Nous y introduisons une nouvelle classe de distributions bivariées de type Marshall-Olkin, la distribution Erlang bivariée. Il est montré que la distribution d'Erlang bivariée a une partie absolument continue et une partie singulière. La transformée de Laplace, les moments et les densités conditionnelles y sont obtenus. Le mélange de ces distributions bivariées Erlang est décrit et les applications potentielles en assurance-vie et en finance sont prises en considération. Les estimateurs du maximum de vraisemblance des paramètres sont calculés par l'algorithme Espérance-Maximisation. Les simulations sont effectuées pour mesurer la performance de l'estimateur.

La deuxième partie, liée à la théorie du risque, est présentée dans les chapitres 4 et 5 de la thèse et est consacrée à l'étude des processus de risque multivariés, qui peuvent être utiles dans l'étude des problèmes de la ruine des compagnies d'assurance avec des classes dépendantes. Nous appliquons les résultats de la théorie des processus de Markov déterministes par morceaux afin d'obtenir les martingales exponentielles, nécessaires pour établir des bornes supérieures calculables pour la probabilité de ruine, dont les expressions sont intraitables.

Comme une extension du modèle de risque multivarié proposé par Asmussen et Albrecher (2010), nous considérons d'abord un processus de risque de dimension m obtenu par la modélisation de la dépendance du nombre de réclamations en utilisant le modèle de Poisson avec chocs communs, ce qui suppose que, en plus des chocs individuels, un choc commun affecte toutes les classes d'affaires d'assurance et qu'un autre choc commun a un impact sur chaque couple de classes. Aussi, la dépendance entre les montants des réclamations et entre les classes est autorisée. Le comportement asymptotique de la probabilité que la ruine survienne simultanément dans toutes les classes avant la période t est étudié.

Inspiré par le travail de Dufresne et Gerber (1991) et de Li, Liu et Tang (2007), nous adoptons l'idée d'ajouter un processus de diffusion caractérisé par un mouvement brownien corrélé de dimension m .

Pour ces deux modèles multivariés, une expression de la probabilité que la ruine se produise dans au moins une classe d'affaires et une borne supérieure pour la probabilité que la ruine survienne simultanément dans toutes les classes sont obtenues. Des résultats numériques pour les bornes supérieures obtenues sont rapportées pour ces modèles en supposant trois classes d'affaires d'assurance, où la dépendance entre les tailles des réclamations est modélisée à l'aide de la notion de copule. Il est établi que l'ajout d'un processus de diffusion conduit à augmenter ces bornes supérieures.

Ensuite, dans un cadre plus réaliste, notre projet de recherche est décrit en enquêtant sur les probabilités de ruine associées à un processus de risque de dimension m qui suppose que, en plus des arrivées de réclamations individuelles pour chaque catégorie d'entreprise régies par des processus de Poisson, il y a des réclamations totales produites par un processus de comptage de renouvellement commun qui affecte toutes les classes d'affaires d'assurance.

Dans ce contexte multivarié, le processus de vecteur surplus devient un processus de Markov en introduisant un processus supplémentaire, et les outils de la théorie des processus de Markov déterministes par morceaux sont appliqués

afin d'obtenir des martingales exponentielles. Sur la base de ces martingales, nous obtenons une borne supérieure pour la probabilité que la ruine se produise dans toutes les classes en même temps. De plus, une borne supérieure pour ce type de probabilité de ruine est obtenue dans un cas particulier où les chocs individuels sont absents et les réclamations à travers les classes ne sont produites que par le processus de renouvellement; cette borne est illustrée par des résultats numériques, où une version bivariée est prise en compte et la dépendance des montants de réclamation est introduite en utilisant des techniques de copules.

Mots-clés: Distribution Erlang, Algorithme Espérance-Maximisation, Processus de Markov déterministes par morceaux, Modèle de risque multivarié, Probabilité de ruine, Modèle de Poisson avec chocs communs, Processus de renouvellement, Copules.

SUMMARY

This dissertation is devoted to modeling dependence with potential applications in actuarial science and is divided in two parts: the first part considers dependence in the context of bivariate survival data analysis and the second, related to risk theory, deals with dependence between classes of an insurance business.

The first part is presented in the form of a research paper in Chapter 3. In this contribution, we introduce a new class of bivariate distributions of Marshall-Olkin type, called bivariate Erlang distributions. It is shown that the bivariate Erlang distribution has both an absolutely continuous and a singular part. The Laplace transform, product moments and conditional densities are derived and also, the finite mixture of the bivariate Erlang distributions is described. Potential applications of bivariate Erlang distributions in life insurance and finance are considered. The maximum likelihood estimators of the parameters are computed via an Expectation-Maximization algorithm. Simulations are carried out to measure the performance of the estimator.

The second part related to risk theory is presented in Chapters 4 and 5 of this thesis and is devoted to the study of multivariate risk processes, which may be useful in analyzing ruin problems for insurance companies with a portfolio of dependent classes of business. We apply results from the theory of piecewise deterministic Markov processes in order to derive exponential martingales needed to establish computable upper bounds for the ruin probabilities, as their exact expressions are intractable.

As an extension of the multivariate risk model proposed by Asmussen and Albrecher (2010), we first consider an m-dimensional risk process obtained by modeling the dependence through the number of claims using the Poisson model

with common shocks. We assume that in addition to the individual shocks, a common shock affects all classes of business and that another common shock has an impact on each pair of classes. Also, dependence between claims sizes across classes is allowed. The asymptotic behavior of the the probability that ruin occurs in all classes simultaneously before a fixed time t, in both cases of dependent heavy-tailed claims and independent heavy-tailed claims, is investigated.

Inspired by the work of Dufresne and Gerber (1991) and of Li, Liu and Tang (2007), we embrace the idea of adding a diffusion process characterized by an m-dimensional correlated Brownian motion.

For each of these two multivariate models an expression for the probability that ruin occurs in at least one class of business and an upper bound for the probability that ruin occurs in all classes simultaneously are derived. Numerical results regarding the upper bounds are reported for these models assuming three classes of insurance business, where the dependence between claims sizes is modeled using the notion of copula. It is established that adding a diffusion process leads to increasing these upper bounds.

Further, in a more realistic setting, our research project is outlined by investigating ruin probabilities associated to an m-dimensional risk process which assumes that in addition to the individual claim arrivals for each class of business, governed by Poisson processes, there are aggregate claims produced by a common renewal counting process that affects all classes of business.

In this multivariate context, the surplus vector process is Markovianized by introducing a supplementary process, and tools from the theory of piecewise deterministic Markov processes are applied in order to obtain exponential martingales. Based on these martingales, we derive an upper bound for the probability that ruin occurs in all classes simultaneously. Also, an upper bound for this type of ruin probability is derived in a special case where the individual shocks are absent and the claims across classes are produced only by the renewal process. The latter upper bound is illustrated by numerical results, where a bivariate version is

considered and the dependence in claim sizes is captured using copula techniques.

Keywords: Erlang distribution, Expectation-Maximization algorithm, Piecewise deterministic Markov processes, Multivariate risk model, Ruin probability, Poisson model with common shocks, Renewal processes, Copulas.

CONTENTS

LIST OF TABLES

ACKNOWLEDGEMENTS

I express my entire gratitude to my thesis advisor Professor Louis Doray for his ongoing support, valuable advice and insightful guidance offered throughout my PhD studies. I am also grateful to my thesis co-advisor Professor Manuel Morales who supervised my work with interest, expert advice and steady encouragements.

I would like to thank the professors from the Department of Mathematics and Statistics of both University of Montréal and Concordia University for their scientific lectures.

Financial support from my supervisors Professor Louis Doray and Professor Manuel Morales, Fonds québécois de recherche sur la nature et les technologies and from the "Micheline Gaudreault" award is gratefully acknowledged.

I would like to thank Professor Mylène Bédard, Professor José Garrido and Professor Emiliano Valdez to have agreed to be members of the evaluation jury for this thesis and to devote their time to evaluate this document.

Many thanks as well to Mrs. Anne-Marie Dupuis, Mrs. Émilie Langlois Dubois and Mr. Michele Nasoni for their help and technical support.

I want to thank my mother Margareta Groparu and my brother Alexandru as well as his family who all have been supportive and caring. This is a moment that I deeply wish my father Stan Groparu could have lived to share.

I will always remember the moments of comfort and joy with my lovely dog Lolita that eased my stress.

I express my appreciation to my husband Cristian for his great support of my professional endeavors and also for providing me with love, understanding and patience along the way. He encouraged me to pursue my dream and I thank him from the bottom of my heart.

INTRODUCTION

Traditionally, actuarial theory of multiple life insurance is based on the assumption of independence for the remaining lifetimes in order to evaluate the premium relating to an insurance contract involving multiple lives. Postulating independence is mathematically convenient and is also triggered by the fact that in general the statistics gathered by the insurer only give information about the marginal distributions of the lifetimes, not about their joint distribution.

However, in many situations this assumption is not valid. Intuitively, pairs of individuals exhibit dependence in mortality because they share common risk factors, which may be purely genetic, as in the case of twins, or environmental, as in the case of a married couple. Several studies have established dependence of the lifetimes of paired lives such as husband and wife [see Parkes et al. (1969), Jagger and Sutton (1991)].

The first actuarial textbook explicitly introducing multiple life models in which the future lifetime random variables are dependent is Bowers et al. (1997).

Therefore, models that incorporate dependence are desirable in the study of multivariate survival data analysis such as familial data, matched pairs, or different components of a system. Several methods for multivariate survival data have been proposed. For an extensive discussion on multivariate models and their properties and applications, we refer to the book by Kotz et al. (2000).

One classical model of dependent lives that captured our attention is called the "common shock" model [Marshall and Olkin (1988), Bowers et al. (1997)]. This model assumes that the lifetimes of two persons, say T_1 and T_2, are independent unless a common shock causes the death of both. For example, a contagious deadly disease, a natural catastrophe or a car accident may affect the lives of the two spouses. Thus, if T_0 denotes the time until the common disaster, the actual ages-at-death are modeled by

$$X = \min(T_1, T_0) \;\text{ and }\; Y = \min(T_2, T_0).$$

In this context, the joint survival function of (X, Y) is given by

$$\begin{aligned}\overline{F}(x, y) &= P\left(X > x, Y > y\right) = P(T_1 > x, T_2 > y, T_0 > \max(x, y)) \\ &= P(T_1 > x)P(T_2 > y)P\left(T_0 > \max(x, y)\right),\end{aligned}$$

in view of the mutual independence of T_0, T_1 and T_2. This model is also called the bivariate survival model of Marshall-Olkin type.

The common shock model turned out to be convenient for annuity valuation purposes because it is easy to implement and suitable for computation; in this sense, we refer to Frees (1996), Frees, Carriere, and Valdez (1996), and Bowers et al. (1997). We mention here that annuities are contractual guarantees that promise to provide periodic income over the lifetime of an individual, called the annuitant.

By assuming that the random variables T_i, $i = 0, 1, 2$, are exponentially distributed, the random vector (X, Y) is said to follow the bivariate exponential distribution, which was proposed by Marshall and Olkin (1967a).

Several other bivariate distributions of Marshall-Olkin type have been proposed. We list some of them namely, the bivariate Weibull distribution suggested by Marshall and Olkin (1967a), the bivariate Pareto distribution [Veenus and Nair (1994)], the bivariate distribution proposed by Al-Khedhairi and El-Gohary (2008), where T_i, $i = 1, 2$, follow Gompertz distributions and T_0 is exponentially distributed, the bivariate distribution derived by Sarhan and Balakrishnan (2007), where T_i, $i = 1, 2$, follow generalized exponential distributions and T_0 follows an exponential distribution and later on, this last bivariate distribution was modified by Kundu and Gupta (2010) assuming different generalized exponential distributions for the components T_i, $i = 0, 1, 2$. We give a detailed presentation of these examples in Section 3.2 of Chapter 3.

Our contribution in this dissertation is to introduce a new class of bivariate distributions, called bivariate Erlang distributions, using Erlang marginals in the survival model of Marshall-Olkin type. The Erlang distribution is a special case of the gamma distribution with non-negative integer shape parameter and further, if the shape parameter equals one, then it reduces to the exponential distribution. Therefore, the bivariate exponential distribution of Marshall and Olkin (1967a) is a particular case of the bivariate Erlang distribution.

Properties of the bivariate Erlang distribution and of the maximum likelihood parameter estimators, using an Expectation-Maximization algorithm, are provided in Chapter 3. Also, the usefulness of these bivariate distributions in finance and insurance is presented in Chapter 3.

Now, we shift our attention to topics from risk theory and their applications in non-life insurance.

In 1930, Harold Cramér wrote that "The object of the theory of risk is to give a mathematical analysis of the random fluctuations in an insurance business and to discuss the various means of protection against their inconvenient effects."

Insurance is a mechanism for transferring risk from one party to another party that is better able to manage it. It is important for insurance companies to set aside an amount of money, as reserves or surplus, in order to meet its commitments and pay claims whenever they occur. To prevent ruin, where the claims paid exceed the reserves available, the company must provide sufficient initial capital and carefully manage its reserves throughout its operations, such as setting premium levels or developing reinsurance strategies.

Mathematically, assuming an initial surplus of u ($u \geq 0$) at time $t = 0$, then the surplus at time t, denoted by $U(t)$, would be

$$U(t) = u + C(t) - S(t),$$

where $\{C(t);\ t \geq 0\}$ is the premium process which measures all premiums collected up to time t, and $\{S(t);\ t \geq 0\}$ is the aggregate claims process, which measures all claims paid up to time t.

The function $C(t)$ should be established in such a way that the solvability of the portfolio can be guaranteed. Usually, it is assumed that $C(t) = ct$, $c > 0$, meaning that the aggregate premium payments are assumed to be collected continuously over time at a constant rate of c per unit of time.

The aggregate claims process is represented as $S(t) = X_1 + X_2 + ... + X_{N(t)}$ (with $S(t) = 0$ if $N(t) = 0$), where $N(t)$ is a process that counts the number of claims up to time t. The individual claim sizes are modeled by X_1, X_2, ..., which are non-negative, independent and identically distributed random variables, with common distribution function $F(x) = P(X \leq x)$ and finite mean $\mu = \int\limits_0^\infty x dF(x)$. It is assumed that $\{X_n\}_{n\geq 1}$ are independent of $N(t)$.

Counting processes used in claim number modeling are for example, Poisson processes, where the claim inter-arrival times are independent and identically distributed, each having an exponential distribution. Other choices include renewal processes, where the claim inter-arrival times do not necessarily follow an exponential distribution but form a sequence of positive, independent and identically distributed random variables.

Claim size distributions are roughly classified into two groups: light-tailed and heavy-tailed distributions. Random variables that tend to assign higher probabilities to large values are said to be heavy-tailed and light-tailed otherwise. Heavy-tailed distributions are more realistic in modeling (large) claim amounts, especially those from general insurance [Rolski et al., (1999)]. It is of interest to actuaries because it is the occurrence (or lack) of large claims that is most influential on profits.

Consequently, the actuary is interested in both the frequency and the size of claims.

The ruin probability, denoted by $\psi(u)$, is the probability that the surplus $U(t)$ ever drops below zero. It may be viewed as an useful indicator of the process security, which is of great importance to risk management. Indeed, the premium rate c should be chosen so that a small $\psi(u)$ results for given u.

Therefore, a task of ruin theory is to search for solutions to the probability of ruin as an explicit function of the initial investment, u, if available, or give approximations or tight bounds otherwise.

Ruin theory for the univariate risk model characterized by the surplus process $U(t)$ has been discussed extensively in the literature. Many results are summarized in the books authored by Asmussen (2000), Rolski et al. (1999), Dickson (2005), Willmot and Lin (2001) or Asmussen and Albrecher (2010).

The piecewise deterministic Markov (PDM) processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models.

Dassios and Embrechts (1989) showed in general how to use this theory for solving insurance risk problems dealing with univariate models. Since then, the martingale technique via PDM processes has become a very systematic approach in dealing with continuous-time risk models. See, for example, Davis (1993), Rolski et al. (1999), Dassios and Jang (2003), Jang (2004, 2007), Liu et al. (2005), Lu et al. (2007), Schmidli (2010).

An insurance portfolio is generally divided into different classes of insurance business and the insureds are classified according to the risk they represent for the insurer, and so the need to introduce multivariate risk processes. For example, typical lines of insurance in a portfolio are health, automobile, liability, or house insurance.

We might say that ruin occurs in a class of business if at some point in the future the surplus becomes negative. As pointed out in Chan et al. (2003), the concept of "ruin" in the multi-dimensional framework could have different meanings and interpretations when compared to the univariate risk process. In this sense, different ruin concepts for multivariate risk processes are introduced by Chan et al. (2003) namely,

• the ruin probability denoted by ψ_{and} is the probability that ruin occurs, not necessarily at the same time, in all classes eventually;

• the ruin probability denoted by ψ_{sim} is the probability that ruin occurs in all classes simultaneously or at the same instant in time;

• the ruin probability denoted by ψ_{or} is the probability that ruin occurs at least in one class of business;

• the ruin probability denoted by ψ_{sum} is the probability that the total surplus of the company is negative for one or more times in the future.

In practice, these types of ruin probabilities can be useful indicators of the portfolio security. If ruin occurs, it is interpreted as if the company has to take action in order to make the business profitable.

In recent years, multivariate risk models have been studied in the literature; for example, Chan et al. (2003), Cai and Li (2005, 2007), Yuen et al. (2006), Dang et al. (2009), Gong et al. (2012), Li et al. (2007). The results of these papers are presented in Subsection 2.4.2.

The purpose of our research project is to examine the multivariate risk processes, which may be useful in studying ruin problems for insurance companies with dependent classes of business. The dependence within the model can have effects on the distribution of the aggregate claims, and consequently, the probability of ruin. The dependence structure incorporated in multivariate models may be defined by introducing interaction between the number of claims and/or between the claims sizes across classes of business.

In general, the properties and expressions of the probabilities of ruin in a multivariate setting are largely unknown, even in a bivariate case.

For this reason, we focus on deriving upper bounds for ruin probabilities.

Our contribution is to reformulate the multivariate risk models of our study in terms of piecewise deterministic Markov (PDM) processes. By employing tools from the PDM processes theory, we derive exponential martingales needed in our ruin problem. More specifically, assuming light-tailed marginal claim size distributions and using these martingales, we obtain Lundberg-type upper bounds for the probability that ruin occurs in all classes simultaneously, denoted by ψ_{sim}.

By adopting this approach, we extend the work of Dassios and Embrechts (1989) to the multivariate risk process.

The first multivariate model we consider for investigation is obtained by modeling the dependence through the number of claims using the Poisson model with common shocks. It assumes that in addition to the individual shocks, a common shock affects the m classes of business and that another common shock has an impact on each couple of classes. Also, dependence between claims sizes across classes of insurance business is allowed. This way, we want to see how the ruin probability is affected by just having independent Poisson processes for claims number processes to then gradually adding common shocks that have impact on couples of classes and finally, on all classes of insurance business. This model would illustrate more realistic situations. For example, a car accident may cause a claim for automobile insurance, health and liability insurance. Moreover, common shocks may affect couples of classes of insurance such as a hurricane or an earthquake will be likely to make claims on both automobile and health policies or on both automobile and homeowner policies.

We note that with this model, we extend the model proposed by Asmussen and Albrecher (2010), where an upper bound for the ruin probability ψ_{sim} was derived assuming that all classes of business share the same Poisson claim number process.

In addition to the upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$, where u_i is the initial capital of the i-th class, for $i = 1, ..., m$, we derive also an expression for the ruin probability $\psi_{or}(u_1, ..., u_m)$.

Inspired by the work of Dufresne and Gerber (1991) and of Li, Liu and Tang (2007), we embrace the idea of adding a correlated m-dimensional Brownian motion to the risk model, where the claims arrivals follow a Poisson model with common shocks. The diffusion process accounts for some of the uncertainty in the aggregate claims or in the premium income.

Supposing that, for $m \geq 3$, the correlation coefficients between the components of the diffusion process are non-negative, except for at most one element, we derive an upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$. This is obtained with the aid of an exponential martingale obtained by using tools from the theory of PDM processes.

The second multivariate model we consider for studying ruin probabilities is a multivariate risk process. It assumes that in addition to the independent underlying risks for each class of business, aggregate claims are produced by a common shock that affects all classes of business.

Assuming that the individual claim arrivals for each class of business are governed by Poisson processes and the claim arrivals due to the common shock are governed by a common renewal process makes this model suitable for realistic situations. It is also more challenging than the aforementioned model, where all common shocks are governed by Poisson processes.

The reason for incorporating a renewal process that counts common shocks, such as natural disasters that affect all classes of business, is due to the fact that the Poisson process does not always give a practical description. More specifically, for a Poisson process the elapsed time since the last event does not influence the timing of the next event to happen; in reality this does not happen. For example, the rate of occurrence of earthquakes, tornadoes, or hurricanes would be affected by climate patterns [Diaz and Murnane (2008)].

In this multivariate setting, we derive upper bounds of Lundberg-type for the ruin probability $\psi_{sim}(u_1, ..., u_m)$, by using martingales obtained with the aid of the theory of PDM processes. Here, we mention that due to the presence of the renewal process, a Markov vector process was obtained by introducing a supplementary variable, namely, $V(t) = t - \sigma_{N(t)}$, which represents the time elapsed since the last renewal claim before time t. This technique, called backward Markovization technique, can be found in Cox (1955).

This thesis is structured as follows.

Chapters 1 and 2 comprise the theoretical background of the dissertation, while Chapters 3, 4 and 5 contain the main results of our study.

Chapter 1 contains a brief review of some of the properties related to the Erlang distribution, mixture of Erlang distributions and the bivariate exponential distribution of Marshall-Olkin type. This provides the right context for the results obtained in Chapter 3.

The aim of Chapter 2 is to review the concept of PDM processes and of martingales, and to provide an introduction to the ideas of ruin theory. Two univariate models extensively analyzed in the actuarial literature are presented, namely the classical risk model (or Cramér-Lundberg model) and its generalization, the renewal model (or Sparre Andersen model). In addition we discuss multivariate ruin models along with results related to bounds, approximations and the asymptotic behavior of the associated ruin probabilities.

In Chapter 3, we define a new bivariate distribution, that we call bivariate Erlang (BVEr) distribution. It is of Marshall-Olkin type's assuming that the random variables T_i, for $i = 0, 1, 2$, follow Erlang distributions.

The BVEr distribution is a mixture of an absolutely continuous distribution and a singular part concentrating its mass on the diagonal line $x = y$, a property that is established in Chapter 3. Also, the joint survival function, probability density function along with the marginal distributions, conditional probability density functions, conditional expectations, and Laplace transform are obtained. Further, this distribution is extended to a bivariate distribution using finite mixtures of Erlang distributions.

Unfortunately, statistical inference for the BVEr distribution is not a simple task due to the complicated nature of its density function. We adopt the Expectation -Maximization (EM) algorithm, which was proposed by Karlis (2003) for the bivariate exponential distribution of Marshall-Olkin type, to compute the maximum likelihood estimators for the bivariate Erlang distribution in the case where the shape parameters are known. The method and simulation results are illustrated in Section 3.7 of Chapter 3.

In Chapter 4, we derive an exponential martingale associated to an m-dimensional risk model, where the claims arrivals are assumed to be dependent Poisson processes with common shocks. With the aid of the tools from PDM processes theory and based on this martingale, the corresponding Lundberg-type upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$ and an expression for the ruin probability $\psi_{or}(u_1, ..., u_m)$ follow. Also, we consider studying the m-dimensional risk model perturbed by diffusion which is characterized by an m-dimensional correlated Brownian motion and an upper bound of the ruin probability $\psi_{sim}(u_1, ..., u_m)$ is obtained.

Numerical results for the upper bounds are provided by assuming a portfolio of three classes of insurance business and employing a member of Farlie-Gumbel-Morgenstern family copula with exponential marginals to model the dependence between claim sizes across classes.

In Chapter 5, we investigate a multivariate risk process that assumes that in addition to the independent underlying risks generated by Poisson processes for each class of business, there are aggregate claims produced by a common shock modeled by an ordinary renewal process that affects all classes of business. Assuming light-tailed marginal claim size distributions, Lundberg-type upper bounds are derived for the the probability that ruin occurs in all classes simultaneously. Also, a special case is treated, where the individual shocks are absent and the claims across classes are generated only by the renewal process.

Numerical results are reported for a special bivariate case based only on common shocks and where the dependence structure between the claim sizes is modeled by the bivariate Farlie-Gumbel-Morgenstern (FGM) copula.

In the final chapter conclusions are drawn and ideas for further research expressed.

Chapter 1

ERLANG AND BIVARIATE EXPONENTIAL DISTRIBUTIONS

In this thesis, we introduce a class of bivariate Erlang distributions along with the finite mixture of these distributions. We show that the bivariate Erlang distribution is an extension of the bivariate exponential distribution proposed by Marshall and Olkin in 1967 [Marshall and Olkin (1967a)].

In order to provide the right context for the results obtained regarding the bivariate Erlang distribution, we start by giving a brief review of some of the properties of the Erlang distribution, the mixture of Erlang distributions and the bivariate exponential distribution of Marshall-Olkin type. The notion of hazard rate is introduced along with related properties.

1.1. Properties of the Erlang distribution

Agner Krarup Erlang [Erlang (1917)] was the first author to extend the exponential distribution with his "method of stages". He defined a non-negative random variable as the time taken to move through a fixed number of stages (or states), spending an exponential amount of time with a fixed rate in each one. Nowadays, we refer to distributions defined in this manner as Erlang distributions.

Definition 1.1.1. *The probability density function (p.d.f.) of the Erlang distribution, denoted by Erlang(k, λ), is given by*

$$f(x) = \begin{cases} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0, \end{cases}$$

where k is a positive integer and λ is a non-negative real number.

The parameter k is called the shape parameter and the parameter λ is called the rate parameter. The distribution is sometimes defined using the inverse of the rate parameter, the scale parameter $\alpha = \lambda^{-1}$.

The Erlang distribution is a continuous distribution with wide applicability primarily due to its relation to the exponential and gamma distributions. When the shape parameter k equals 1, the distribution simplifies to the exponential distribution. The Erlang distribution is a special case of the gamma distribution where the shape parameter k is restricted to the integers.

The following lemma gives some properties of the Erlang distribution.

Lemma 1.1.1. *(1) If $X \sim$ Erlang(k, λ), then the survival function of X is*

$$\overline{F}(x) = P(X > x) = e^{-\lambda x} \sum_{n=0}^{k-1} \frac{(\lambda x)^n}{n!}, \quad x > 0.$$

(2) If $X \sim$ Erlang(k, λ), then the mean of X is $E[X] = k/\lambda$ and the variance of X is $Var[X] = k/\lambda^2$.

(3) If $X \sim$ Erlang(k, λ), then $aX \sim$ Erlang$(k, \lambda/a)$, for $a > 0$.

(4) If X and Y are independent random variables with $X \sim$ Erlang(k_1, λ), $Y \sim$ Erlang(k_2, λ), then $X + Y \sim$ Erlang$(k_1 + k_2, \lambda)$.

(5) If X_1,...,X_k are independent and identically distributed (i.i.d.) random variables with $X_i \sim$ Exponential(λ), then $\sum_{i=1}^{k} X_i \sim$ Erlang(k, λ).

For a proof of these properties, we refer to Cox (1967).

The Erlang distribution was developed by A.K. Erlang to examine the number of telephone calls which might be made at the same time to the operators of the switching stations. This work has been expanded to consider waiting times in queuing systems in general. That is, when events occur independently with some average rate and are modeled with a Poisson process, the waiting times between k occurrences of the event are Erlang distributed. The Erlang distribution is one of the most commonly used distributions in queuing theory which is closely related to risk theory; see for example, Asmussen (1987, 1989) and Takács (1962).

1.2. Mixture of Erlang distributions

We start by defining a mixture of densities in general terms as follows.

Definition 1.2.1. *A parametric family of mixture densities is a family of probability density functions of the form*

$$f(x;\Phi) = \sum_{i=1}^{\infty} \alpha_i f_i(x;\phi_i), \quad x = (x_1, ..., x_n) \in \mathbb{R}^n, \tag{1.2.1}$$

where each α_i is nonnegative and $\sum_{i=1}^{\infty} \alpha_i = 1$, each f_i is itself a density function parametrized by $\phi_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$, and $\Phi = (\alpha_i, \phi_i, i = 1, 2, ...)$. If there exists a non-negative integer m such that $\alpha_i = 0$ for $i > m$, then (1.2.1) *defines a family of finite mixture densities.*

Finite mixture densities arise naturally as densities associated with a statistical population which is a mixture of m component populations, with associated component densities $\{f_i\}$, $i = 1, 2, ..., m$, and mixing proportions $\{\alpha_i\}$, $i = 1, 2, ..., m$. Such densities often are of interest in life testing experiments such as testing systems, devices, recording failure times or causes [cf. Mendenhall and Hader (1958), Cox (1959)].

Mixture of Erlang distributions are obtained by replacing the densities $f_i(x;\phi_i)$, $i = 1, 2, ...$ in (1.2.1) with the densities of Erlang distributions, namely

$$f_i(x;\lambda_i,k_i) = \frac{\lambda_i^{k_i} x^{k_i-1} e^{-\lambda_i x}}{(k_i-1)!}, \quad \text{for} \quad x > 0,$$

where k_i are positive integers and λ_i are non-negative real numbers, for $i \geq 1$.

Mixture of Erlang distributions with a common rate parameter, say λ, define a very broad parametric distribution class since any absolutely continuous distribution on $(0,\infty)$ may be approximated arbitrarily accurately by a distribution of this type [Tijms (1994), pp. 163-164]. More specifically, for any positive continuous distribution with density $g(x)$ and distribution function $G(x)$, the distribution function defined by the density function

$$\widehat{g}(x) = \sum_{j=1}^{\infty} \left[G\left(\frac{j}{\lambda}\right) - G\left(\frac{j-1}{\lambda}\right) \right] \frac{e^{-\lambda x} \lambda^j x^{j-1}}{(j-1)!}, \quad x > 0, \tag{1.2.2}$$

converges to $G(x)$ pointwise, as $\lambda \to \infty$.

In order to have a complete image regarding mixtures of Erlang distributions, we conclude this section by presenting a class of multivariate Erlang mixtures with common rate parameter, proposed by Lee and Lin (2012).

Definition 1.2.2. *The density of a p-variate Erlang mixture is defined as*

$$g(x_1, ..., x_p; \lambda, \alpha) = \sum_{k_1=1}^{\infty} ... \sum_{k_p=1}^{\infty} \alpha_{k_1}...\alpha_{k_p} \prod_{j=1}^{p} f_j(x_j; k_j, \lambda), \tag{1.2.3}$$

where $\alpha = (\alpha_{k_1}, ...\alpha_{k_p},\ k_i = 1, 2, ...,\ i = 1, 2, ..., p)$ *with each* $\alpha_{k_1}, ..., \alpha_{k_p} \geq 0$ *and* $\sum\limits_{k_1=1}^{\infty} ... \sum\limits_{k_p=1}^{\infty} \alpha_{k_1}...\alpha_{k_p} = 1$, *and for each* $j = 1, ..., p$, $f_j(x_j; k_j, \lambda) = \frac{\lambda^{k_j} x_j^{k_j-1} e^{-\lambda x_j}}{(k_j-1)!}$, $x_j > 0$, *is the density of the Erlang distribution with shape parameter* k_j *and rate parameter* λ.

Therefore, a distribution in this class is a mixture such that each of its component distributions is the joint distribution of independent Erlang distributions that share a common rate parameter. For this class of multivariate Erlang mixtures, Lee and Lin (2012) derived the following property.

Proposition 1.2.1. *The class of multivariate Erlang mixtures of the form in* (1.2.3) *is dense in the space of positive continuous multivariate distributions in the sense of weak convergence.*

Also, they showed that this class of multivariate Erlang mixtures could be an ideal multivariate parametric model for insurance modeling.

1.3. Failure rate

In Chapter 3, we discuss about possible applications of the bivariate Erlang distribution in reliability theory. In this regard, we mention that in reliability theory, classes of distribution functions are introduced to study lifetimes of systems, devices or components. These distribution functions are often characterized in terms of failure rates and examples of classes of life distributions include the increasing failure rate (IFR) class, or decreasing failure rate (DFR) class. In this section, we recall the definitions of these concepts.

Definition 1.3.1. *Consider a positive random variable* X *with cumulative distribution function* $F(x) = P(X \leq x)$, $x > 0$, *and survival function* $\overline{F}(x) = P(X > x) = 1 - F(x)$. *Suppose that* F *is absolutely continuous with probability density* *function $f(x) = F'(x)$. Then, the failure rate, $r(x)$, of X, also called hazard rate, is defined as*

$$r(x) = \frac{f(x)}{\overline{F}(x)} = -\frac{d \ln \overline{F}(x)}{dx}. \tag{1.3.1}$$

The random variable X may represent the time-until-death of an individual, or the amount of an insurance loss. Expression (1.3.1) can be interpreted as the rate at which an x-year old object will fail. In life insurance, the hazard function is known as the force of mortality [Bowers et al. (1997), Chapter 3].

Definition 1.3.2. *The distribution function F is said to be decreasing (increasing) failure rate or DFR (IFR) if $\frac{\overline{F}(x+y)}{\overline{F}(x)}$ is nondecreasing (nonincreasing) in x for fixed $y \geq 0$; that is, $\overline{F}$ is log-convex (log-concave).*

Consequently, we have that if F is absolutely continuous, then the distribution function F is DFR (IFR) if and only if the failure rate $r(x)$ is nonincreasing (nondecreasing) in x.

In what follows, we present some examples that will be used in this thesis.

Example 1.3.1. *The Erlang distribution is increasing failure rate for $k \geq 2$. For $k = 1$, the Erlang distribution becomes the exponential distribution having the failure rate equal to the constant λ.*

Example 1.3.2. *Consider the following mixture of Erlang distributions with the same rate parameter λ given by*

$$f(x) = \sum_{k=1}^{r} a_k f_k(x) = \sum_{k=1}^{r} a_k \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!},$$

where $a_k \geq 0$ for $k = 1, 2, ..., r$ are the mixing proportions with $a_1 + ... + a_r = 1$. If we denote by $A_k = a_{k+1} + a_{k+2} + ... + a_r$, with $A_0 = 1$, then the corresponding survival function is

$$\overline{F}(x) = \sum_{k=1}^{r} a_k \overline{F}_k(x) = e^{-\lambda x} \sum_{k=1}^{r} a_k \sum_{j=0}^{k-1} \frac{(\lambda x)^j}{j!} = e^{-\lambda x} \sum_{j=0}^{r-1} A_j \frac{(\lambda x)^j}{j!}.$$

We shall denote by $E_{12\ldots r}$ *the above mixture of Erlang distributions of order* r*. Its failure rate function is*

$$r(x) = \frac{f(x)}{\overline{F}(x)} = \lambda \frac{\sum_{j=0}^{r-1} a_{j+1} \frac{(\lambda x)^j}{j!}}{\sum_{j=0}^{r-1} A_j \frac{(\lambda x)^j}{j!}} = \lambda \left[1 - \frac{\sum_{j=0}^{r-1} A_{j+1} \frac{(\lambda x)^j}{j!}}{\sum_{j=0}^{r-1} A_j \frac{(\lambda x)^j}{j!}} \right].$$

It has been shown in Willmot and Lin (2001) that E_{12} *is always increasing failure rate, while for* E_{123}*, the failure rate is first decreasing and then becomes increasing.*

For more discussions on the failure rate of the mixture of Erlang distributions, we refer to Esary et al. (1973) or Willmot and Lin (2001).

1.4. Bivariate exponential distribution of Marshall-Olkin type

In Chapter 3, for the bivariate Erlang distribution we derive properties such as moments, the Laplace transform, the covariance and the correlation structure. This motivates the presentation in this section of similar properties for the bivariate exponential distribution of Marshall-Olkin type, which is a particular case of the bivariate Erlang distribution.

Marshall and Olkin (1967a) proposed the bivariate exponential (BVE) distribution, and a generalization considered also in a further paper [Marshall and Olkin (1967b)].

Definition 1.4.1. *A random vector* (X, Y) *is said to have a bivariate exponential (BVE) distribution of Marshall-Olkin type if its joint survival function is defined as*

$$\overline{F}_{X,Y}(x, y) = P(X > x, Y > y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x,y)}, \quad x, y > 0, \qquad (1.4.1)$$

where $0 < \lambda_i < \infty$*,* $i = 1, 2$*, and* $0 \leq \lambda_0 < \infty$*.*

An interpretation of the BVE distribution is based on shock failures. More specifically, the BVE distribution is obtained by supposing that failure is caused by three types of shocks on a system containing two components, say A and B. These shocks occur independently according to Poisson processes with intensity parameters λ_1, λ_2 and λ_0, respectively (in Subsection 2.2.1.1, the Poisson process and its properties are reviewed). The first (second) type of shock affects only the component A (B), while the third type of shock causes the failure of both A and B, so that their lifetimes X and Y will be dependent when $\lambda_0 > 0$. Then (1.4.1) gives the joint distribution function of the lives of the two components.

For more discussions on the BVE distribution and its applications in reliability theory, we refer to Basu and Block (1975) or Galambos and Kotz (1978).

The BVE distribution is not absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}_2^+, \mathfrak{B}(\mathbb{R}_2^+))$, where $\mathbb{R}_2^+$ is the positive orthant of the (x, y) plane and $\mathfrak{B}(\mathbb{R}_2^+)$ is the corresponding Borel σ-field, having $P(X = Y) = \frac{\lambda_0}{\lambda_1+\lambda_2+\lambda_0} > 0$ for $\lambda_0 > 0$. In this sense, Marshall and Olkin (1967a) derived the following result.

Proposition 1.4.1. *If $\overline{F}_{X,Y}(x, y)$ is BVE(λ_1, λ_2, λ_0) given by* (1.4.1)*, then*

$$\overline{F}_{X,Y}(x, y) = \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \lambda_0}\overline{F}_{ac}(x, y) + \frac{\lambda_0}{\lambda_1 + \lambda_2 + \lambda_0}\overline{F}_s(x, y),$$

where

$$\overline{F}_s(x, y) = e^{-(\lambda_1+\lambda_2+\lambda_0)\max(x,y)}$$

is a singular distribution, and

$$\overline{F}_{ac}(x, y) = \frac{\lambda_1 + \lambda_2 + \lambda_0}{\lambda_1 + \lambda_2}\overline{F}_{X,Y}(x, y) - \frac{\lambda_0}{\lambda_1 + \lambda_2}e^{-(\lambda_1+\lambda_2+\lambda_0)\max(x,y)}$$

is absolutely continuous.

Bhattacharyya and Johnson (1971) constructed a dominating measure as a mixture of one- and two-dimensional Lebesgue measures, with respect to which the probability measure determined by (1.4.1) is absolutely continuous. This measure denoted by μ is defined as

$$\mu(A) = \mu_2(A) + \mu_1((A \cap \{x = y\})_p) \text{ for } A \in \mathfrak{B}(\mathbb{R}_2^+),$$

where μ_i $(i = 1, 2)$ denotes i-dimensional Lebesgue measure, and the subscript p denotes the projection of the set in question onto the x-axis.

In the sequel, properties of the bivariate exponential distribution of Marshall-Olkin type are presented.

Proposition 1.4.2. *Assume that the vector (X, Y) follows a BVE distribution defined by* (1.4.1)*. Then,*

(1) *The marginals X and Y are exponentially distributed with parameters $\lambda_1 + \lambda_0$ and $\lambda_2 + \lambda_0$, respectively.*

(2) $\min(X,Y)$ *is exponentially distributed with parameter* $\lambda_0+\lambda_1+\lambda_2$.

(3) (X,Y) *is BVE if and only if there exist independent exponential random variables* U, V *and* W *such that* $X=\min(U,W)$, $Y=\min(V,W)$.

(4) *The Laplace transform is described as*

$$L_{X,Y}(r,t)=E[e^{-rX-tY}]=\frac{(\lambda_1+\lambda_2+\lambda_0+r+t)(\lambda_1+\lambda_0)(\lambda_2+\lambda_0)+rt\lambda_0}{(\lambda_1+\lambda_2+\lambda_0+r+t)(\lambda_1+\lambda_0+r)(\lambda_2+\lambda_0+t)}.$$

(5) *The covariance and the correlation of* (X,Y) *are given by*

$$Cov(X,Y)=\frac{\lambda_0}{(\lambda_1+\lambda_2+\lambda_0)(\lambda_1+\lambda_0)(\lambda_2+\lambda_0)},\ \ and$$

$$Corr(X,Y)=\frac{\lambda_0}{\lambda_1+\lambda_2+\lambda_0}\in[0,1],$$

respectively. Thus, the random variables X *and* Y *are independent if and only if they are uncorrelated.*

For the proof of these properties, we refer to Marshall and Olkin (1967a).

The following proposition establishes the lack of memory property of the exponential distribution, property that is applied in ruin problems, as we will see in Chapter 2.

Proposition 1.4.3. *If* X *is exponentially distributed with rate parameter* λ, *then*

$$P(X>s+t\mid X>s)=P(X>t), \tag{1.4.2}$$

for all $s\geq 0$, $t\geq 0$.

Moreover, the exponential distribution is the only univariate continuous distribution that exhibits the memoryless property (1.4.2).

For the proof, we refer to Rolski et al. (1999).

Similar to the univariate exponential distribution, the BVE distribution also satisfies the lack of memory property which is illustrated by the following proposition.

Proposition 1.4.4. *If* (X,Y) *has a BVE distribution defined by* (1.4.1), *then*

$$P(X>s_1+t,Y>s_2+t\mid X>s_1,Y>s_2)=P(X>t,Y>t), \tag{1.4.3}$$

for all $s_1\geq 0$, $s_2\geq 0$, $t\geq 0$.

Moreover, the BVE distribution defined by (1.4.1) *is the only bivariate distribution that has exponential marginals and which satisfies the lack of memory property described by* (1.4.3).

For the proof, we refer to Marshall and Olkin (1967a).

An interpretation of property (1.4.3) is that for a two-component system with functioning components of ages s_1 and s_2, the probability that both components will be functioning t time units from now is the same as if both components were new.

We conclude this section by mentioning that Marshall and Olkin (1967a) also derived a natural extension of the BVE distribution to a multivariate distribution with exponential marginals that fulfills a multivariate lack of memory property.

Chapter 2

RUIN MODELS

Ruin theory provides stochastic models of the so-called surplus (or reserve) process, which represents the net value of an insurance portfolio of policies throughout time. In a technical sense, we might say that ruin occurs if at some point in the future the net value of the portfolio becomes negative. The probability of this event is called the probability of ruin, and it is often used as a measure of security for a portfolio. Initial reserves, incoming premiums, the aggregate claims that are made on a portfolio or collection of policies are among the factors taken into account in studying the behavior of the surplus process over time.

An insurance portfolio is generally divided in different classes and the insureds are classified according to the risk they represent for the insurer. For this reason, great interest has been shown in developing multivariate ruin models. Multi-dimensional risk theory gained a lot of attention in the past few years mainly due to the complexity of the problems and the lack of closed form results even under very basic model assumptions.

In this dissertation, we continue the search for new results in the field of multivariate risk processes and focus on deriving bounds and asymptotics for the associated multivariate ruin probabilities, which are usually intractable.

The purpose of this chapter is to present some global characteristics of the surplus process and to give a brief review of some of the univariate and multivariate ruin models associated to the surplus process and the surplus vector process respectively, along with results related to bounds and asymptotic behavior of the ruin probabilities.

Throughout this thesis, we adopt the following notations:

- $f(x) \sim g(x)$ as $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$;
- $f(x) = o(g(x))$ as $x \to x_0$ means $\lim_{x\to x_0} f(x)/g(x) = 0$, where x_0 is any number, or ∞, or $-\infty$;
- $f(x) = O(g(x))$ as $x \to x_0$, means that there exists a constant C and a neighborhood Δ of x_0 such that for all $x \in \Delta$, $|f(x)| \leq C|g(x)|$, where x_0 is any number, or ∞, or $-\infty$.

In order to provide a better understanding of the results presented in this thesis, we begin by reviewing the concept of piecewise deterministic Markov processes and of martingales.

2.1. Piecewise deterministic Markov (PDM) processes and martingales

This section explains the basic definition of a piecewise deterministic Markov (PDM) process that is adopted from Dassios and Embrechts (1989). They focused on those aspects of the definition which are important for applications in risk theory. A detailed, mathematical discussion can be found in Davis (1984) who introduced the class of PDM processes.

We start by giving the definition of a Markov process.

Throughout this thesis we will always use a probability space $(\Omega,\ \mathcal{F},\ P)$ on which all stochastic quantities are defined.

Definition 2.1.1. *A filtration on $(\Omega, \mathcal{F}, P)$, denoted by $\{\mathcal{F}_t\}_{t\geq 0}$, is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$: for any $0 \leq s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.*

A probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration is called a filtered probability space.

An event $A \in \mathcal{F}_t$ is an event such that given the information $\mathcal{F}_t$ at time t, the observer can decide whether A has occurred or not. A process whose value at time t is revealed by the information $\mathcal{F}_t$ is said to be adapted.

Definition 2.1.2. *A stochastic process $X = \{X(t), t \geq 0\}$ is said to be adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ or $\mathcal{F}_t$-adapted if, for each $t \geq 0$, the random variable $X(t)$ is $\mathcal{F}_t$-measurable.*

If the only observation available is the past values of a process X, then the information is represented by the natural filtration of X defined as follows:

Definition 2.1.3. *The natural filtration or the history of a process $X = \{X(t), t \geq 0\}$ is the filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$, where $\mathcal{F}_t^X$ is the σ-algebra generated by the past values of the process:*

$$\mathcal{F}_t^X = \sigma\left(X(s), 0 \leq s \leq t\right).$$

Assume that $X = \{X(t), t \geq 0\}$ is a continuous-time stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ taking values in a state space $(E, \mathfrak{B}(E))$, where $\mathfrak{B}(E)$ is the Borel σ-algebra in E and $E \subseteq \mathbb{R}^d$.

Definition 2.1.4. *A function $p : [0,\infty) \times E \times [0,\infty) \times \mathfrak{B}(E) \to [0,1]$ is a Markov transition measure provided*

1. $p(\cdot,\cdot,\cdot,A)$ is measurable as a function of (s,x,t), for each $A \in \mathfrak{B}(E)$;

2. $p(s,x,t,\cdot)$ is a probability measure on $\mathfrak{B}(E)$ for all $s,t \geq 0$ and $x \in E$; when integrating a function f with respect to this measure we write $\int f(y)p(s,x,t,dy)$;

3. for all $A \in \mathfrak{B}(E)$, $x \in E$ and $s,t > 0$,

$$p(s,x,t,A) = \int p(s,x,u,dy)p(u,y,t,A).$$

Note that $p(s,x,t,A)$ is the probability that the process takes a value in A at time t, if it is started at the point x at time s.

Definition 2.1.5. *An adapted process $\{X(t), t \geq 0\}$ is a Markov process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, if for all $t \geq s$ and Borel sets $A \in \mathfrak{B}(E)$:*

$$P\left(X(t) \in A \mid \mathcal{F}_s\right) = P\left(X(t) \in A \mid X(s)\right).$$

Moreover, a Markov process is said to be time-homogeneous if the transition probability depends only on $t - s$, and not the individual times.

Thus, the Markov property of a process means that the only information relevant to evaluating the behavior of the process beyond time s is the value of the current state $X(s)$.

Proposition 2.1.1. *If the stochastic process X has independent increments, that is, for all $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n < \infty$, the increments $X(t_0) - X(0)$, $X(t_1) - X(t_0)$,...,$X(t_n) - X(t_{n-1})$ are independent random variables, then it is a Markov process. In addition, if the process has stationary increments, in the* *sense that for $0 < s < t$, the distribution of $X(t) - X(s)$ depends only on the length $(t - s)$ of the time interval $[s, t]$, then it is a Markov process homogeneous in time.*

The proof of this proposition can be found in Kannan (1979).

Associated with the Markov process $X(t)$, is the infinitesimal generator $\mathcal{A}$, which is defined as follows.

Definition 2.1.6. *The infinitesimal generator of a (time-homogeneous) Markov process $X(t)$ acting on a function $f(X(t))$ belonging to its domain $D(\mathcal{A})$ is*

$$\mathcal{A}f(X(t)) = \lim_{h \downarrow 0} \frac{E[f(X(t+h)) \mid X(t) = x] - f(X(t))}{h}, \tag{2.1.1}$$

where the domain $D(\mathcal{A})$ is the set of all measurable functions f for which the above limit exists and $E[f(X(t+h)) \mid X(t) = x] = \int\limits_E f(y)p(t+h, x, dy)$.

In other words, $\mathcal{A}f(X(t))$ is the expected increment of the process $f(X(t))$ between t and $t+h$, given the history of $X(t)$ at time t. From this interpretation, the following inversion formula is obtained:

$$E[f(X(t+h)) \mid X(t) = x] - f(X(t)) = \int_0^h E[\mathcal{A}f(X(s))]ds, \tag{2.1.2}$$

which is Dynkin's formula. For more details, we refer to Dynkin (1965).

We now introduce piecewise deterministic Markov (PDM) processes, essentially following the presentation in Dassios and Embrechts (1989). As the name suggests, the evolution of these processes is non-random between random jumps. The deterministic evolution can be described in terms of vector fields, while the jumps are determined by a jump intensity. A formal definition of PDM processes follows.

Definition 2.1.7. *A PDM process is a Markov process $\{X(t), t \geq 0\}$ with two components (η_t, ξ_t):*

- *η_t takes values in a countable set K, labelling the evolution of the process through different stages (for example, $K = \{0, 1\}$ where 0 denotes ruin and 1 denotes non-ruin) and*
- *given $\eta_t = n \in K$, ξ_t takes values in an open set $M_n \subset \mathbb{R}^{d(n)}$ for some function $d : \mathbb{N} \to \mathbb{N}$.*

The state space of $X(t)$ is equal to $E = \{(n,z) : n \in K, z \in M_n\}$. We further assume that for every point $x = (n,z) \in E$, there is a unique, deterministic integral curve $\phi_n(t,z) \subset M_n$, determined by a differential operator χ_n on $\mathbb{R}^{d(n)}$, such that $z \in \phi_n(t,z)$.

If for some $t_0 \in [0,\infty)$, $X(t_0) = (n_0, z_0) \in E$, then ξ_t, with $t \geq t_0$, follows $\phi_{n_0}(t,z_0)$ until either $t = \sigma_0$, where σ_0 is some random time with hazard rate function λ, or until $\xi_t \in \partial M_{n_0}$, the boundary of M_{n_0}. In both cases, the process $X(t)$ jumps, according to a Markov transition measure p on E, to a point $(n_1, z_1) \in E$. ξ_t again follows the deterministic path $\phi_{n_1}(t,z_1)$ until a random time σ_1 (independent of σ_0) or until $\xi_t \in \partial M_{n_1}$, and so forth. The jump times σ_i are assumed to satisfy the following condition:

$$E\left[\sum_i I(\sigma_i \leq t)\right] < \infty \text{ for any } t > 0. \tag{2.1.3}$$

In regard with applications of these processes in ruin models, it is important to have an explicit form of the infinitesimal generator $\mathcal{A}$, as we will see in Subsection 2.3.4. Davis (1984) formulated the following result:

Proposition 2.1.2. *Let $\{X(t), t \geq 0\}$ be a PDM process defined by the transition measure p and the differential operator χ. Then, the infinitesimal generator $\mathcal{A}$ of $X(t)$, defined by* (2.1.1), *has the following expression*

$$\mathcal{A}f(x) = \chi f(x) + \lambda \int_E [f(y) - f(x)] p(x; dy) \text{ for any } f \in D(\mathcal{A}). \tag{2.1.4}$$

In some cases, it is important to have time t as an explicit component of the PDM process, and therefore, $\mathcal{A}$ can be decomposed as $\partial/\partial t + \mathcal{A}_t$, where $\mathcal{A}_t$ is given by (2.1.4) *with possibly time-dependent coefficients.*

The following proposition gives a sufficient criterion for the membership in the domain of $\mathcal{A}$; it can be found in Davis (1984).

Proposition 2.1.3. *Consider that $\{X(t), t \geq 0\}$ is a PDM process defined by the transition measure p. Let Γ be the set of boundary points of E, $\Gamma = \{(n,z) : n \in K, z \in \partial M_n\}$, and let $\mathcal{A}$ be an operator acting on measurable functions $f : E \cup \Gamma \to \mathbb{R}$ satisfying the following:*

(i) The function $t \to f(n, \phi_n(t,z))$ is absolutely continuous for $t \in [0, t(n,z)]$ for

all $(n,z) \in E$*;*

(ii) For all $x \in \Gamma$*,* $f(x) = \int_E f(y)p(x;dy)$ *(boundary condition);*

(iii) For all $t \geq 0$*,* $E[\sum_{\sigma_i \leq t} |f(X(\sigma_i)) - f(X(\sigma_i -))|\ ] < \infty$*,* $X(\sigma_i -) = \lim_{t \uparrow \sigma_i} X(t)$*.*

Then, the set of measurable functions satisfying (i), (ii), and (iii) form a subset of the domain of the generator $\mathcal{A}$*, denoted by* $D(\mathcal{A})$*. Also, in view of* (2.1.3)*, condition (iii) is satisfied if* f *is bounded.*

There is a close relationship between Markov processes and martingales. Before considering this, we shall need some basic facts about martingales.

Definition 2.1.8. *The process* $X = \{X(t), t \geq 0\}$ *is a martingale with respect to the filtration* $\{\mathcal{F}_t\}_{t \geq 0}$ *if*

1. $X(t)$ *is* $\mathcal{F}_t$*-measurable for* $t \geq 0$*;*
2. $E[|\ X(t)\ |\ ] < \infty$ *for all* $t \geq 0$*;*
3. *With probability one,* $E[X(t) \mid \mathcal{F}_s] = X(s)$ *for* $0 \leq s \leq t$*.*

Intuitively, a martingale is a process where the current state $X(s)$ is always the best prediction for its future states.

Given the information in $\mathcal{F}_t$, if one can determine whether the event has happened $(\tau \leq t)$ or not $(\tau > t)$, then the random time τ is called a stopping time. More formally, we have the following definition.

Definition 2.1.9. *A random variable* $\tau : \Omega \to [0,\infty]$ *is called a stopping time for* X *if* τ *is* $\mathcal{F}_t$*-adapted for any* $t \geq 0$*, that is,* $(\tau \leq t) \in \mathcal{F}_t$*,* $t \geq 0$*.*

Stopping times can be thought of as the time when a given event occurs. If it has the value $\tau = \infty$, then the event never occurs.

Example 2.1.1. *The hitting time of an open set* A *which is defined by the first time when* X *reaches* A*:* $\tau_A = \inf\{t \geq 0 : X(t) \in A\}$ *is a stopping time for* X*.*

Example 2.1.2. *Every deterministic time* $t \geq 0$ *is a stopping time and in addition, if* τ *is a stopping time, so is* $t \wedge \tau = \min(t,\tau)$*.*

The following result known as the "Optional Stopping Property" is essential for our applications and its proof can be found in Kannan (1979).

Proposition 2.1.4. *Let* $X(t)$ *be a martingale with respect to a suitable filtration and* τ *be a stopping time. If* $P(\tau < \infty) = 1$ *and* $E[\sup_{t \geq 0} |X(t)|\ ] < \infty$ *(uniform integrability), then* $E[X(\tau)] = E[X(0)]$*.*

An application of Dynkin's formula (2.1.2) provides the following result from Davis (1984), giving a connection between Markov processes and martingales.

Proposition 2.1.5. *Let $\{X(t), t \geq 0\}$ be a homogeneous Markov process with generator $\mathcal{A}$ and assume that the martingales will always be with respect to the natural filtration $\sigma(X_s, 0 \leq s \leq t)$.*

(1) If for all t, $f(\cdot, t)$ belongs to the domain of $\mathcal{A}_t$ and $(\partial/\partial t)f(x,t)+\mathcal{A}_t f(x,t) = 0$, then the process $f(X(t), t)$ is a martingale.

(2) If f belongs to the domain of $\mathcal{A}$ and $\mathcal{A}f(x) = 0$, then $f(X(t))$ is a martingale.

In our study, Proposition 2.1.5 will be used in order to derive martingales needed in ruin related problems.

A characterization of the surplus process followed by a review of some of the univariate and multivariate ruin models complete this chapter.

2.2. The surplus process

Given a particular portfolio, the surplus process, denoted by $\{U(t), t \geq 0\}$, is defined as a model for the evolution in time of the reserves of an insurance company. In the sequel, we present the structure of this model.

- The claim arrival times (epochs), denoted by σ_n, are random times at which claims occur. With probability one, $0 = \sigma_0 < \sigma_1 < \sigma_2 < ....$

- The claim number process, denoted by $\{N(t), t \geq 0\}$, is defined by the number of claims in the interval $[0, t]$:

$$N(t) = \max\{n \geq 0 : \sigma_n \leq t\} = \min\{n \geq 0 : \sigma_{n+1} > t\},$$

which leads to

$$\{N(t) = n\} = \{\sigma_n \leq t < \sigma_{n+1}\}. \tag{2.2.1}$$

It is assumed there are only finitely many claims in finite time intervals.

- The claim inter-arrival times, denoted by T_n, $n = 1, 2, ...$, are positive random variables, where T_1 is the time until the first claim and for $n > 1$, T_n is the time between the $(n-1)$-th claim and n-th claim. Hence,

$$\sigma_n = T_1 + ... + T_n, \quad \text{for} \quad n \geq 1.$$

- The size of the n-th claim is denoted by X_n and it is assumed that X_n, $n = 1, 2, ...$, are positive independent and identically distributed (i.i.d.) random variables having common distribution function F, with $F(0) = 0$, and mean value $\mu = E[X_n] = \int_0^\infty [1 - F(x)]dx > 0$.

• The premium payments are assumed to be collected continuously over time at a constant rate of c ($c > 0$) per unit time.

• The initial surplus is denoted by $u = U(0) \geq 0$.

Given this structure, the resulting risk process is introduced as follows.

Definition 2.2.1. *The surplus $U(t)$ at time t is defined as*

$$U(t) = u + ct - S(t),$$

where the process $\{S(t), t \geq 0\}$ is called the aggregate claims process and is given by $S(t) = \sum_{k=1}^{N(t)} X_k$. By convention, $S(t) = 0$ if $N(t) = 0$.

This is the standard model of an insurance company: at each point of $N(t)$ the company has to pay out a stochastic amount of money, and the company receives (deterministically) c units of money per unit time.

For a given surplus process, the following quantities may be useful indicators of the process security and relevant for various insurance-related problems.

Definition 2.2.2. *The time of ruin is defined as*

$$\tau = \tau(u) = \inf\{t \geq 0 : U(t) < 0 | U(0) = u\},$$

and represents the first time the surplus of the insurance company with an initial capital of u goes below zero. If the process $U(t)$ never assumes a negative value (no ruin occurs), we indicate this writing $\tau = \infty$.

Definition 2.2.3. *The ruin probability with finite time horizon t is defined as*

$$\psi(u, t) = P(\tau \leq t | U(0) = u),$$

and is the probability that the ruin happens before time t.

Definition 2.2.4. *The infinite horizon (time) ruin probability is defined as*

$$\psi(u) = P(\tau < \infty | U(0) = u),$$

and is the probability that the surplus ever drops below zero when the initial surplus is u. We shall omit the adjective "infinite horizon".

Note that $\psi(u) = 1$ for $u < 0$, $\psi(u, t)$ is increasing in t and decreasing in u, and $\lim_{t \to \infty} \psi(u, t) = \psi(u)$.

Definition 2.2.5. *The survival probability is $\phi(u) = 1 - \psi(u)$.*

The surplus process from Definition 2.2.1 can also be written as

$$U(t) = u + \sum_{k=1}^{n}(cT_k - X_k) + c(t - \sum_{k=1}^{n} T_k),$$

where by (2.2.1), t is such that $\sum_{k=1}^{n} T_k \leq t < \sum_{k=1}^{n+1} T_k$. As a consequence of the assumption that $c > 0$, ruin may occur when t is a claim epoch and therefore, the ruin probability $\psi(u)$ is equivalent to

$$\begin{aligned} \psi(u) &= P\left(\sum_{k=1}^{n}(X_k - cT_k) > u \text{ for some } n = 1, 2, ...\right) \\ &= P\left(\sup_{n\geq 0} Z_n > u\right), \end{aligned} \tag{2.2.2}$$

where the random variables Z_n, $n = 0, 1, 2, ...$, are defined by

$$Z_0 = 0 \text{ and } Z_n = \sum_{k=1}^{n}(X_k - cT_k) \text{ for } n \geq 1. \tag{2.2.3}$$

The initial surplus u is interpreted as the initial capital the company is willing to risk. The ruin probability is a function of u and is of great interest to preventive maintenance and risk management. For instance, the premium rate c should be chosen so that a small $\psi(u, t)$ results for a given u and t, or a small $\psi(u)$ results for given u.

Since financial ruin will happen with probability one if $P(\liminf_{t\uparrow\infty} U(t) = -\infty) = 1$, an additional assumption imposed on the models is that $U(t) \to \infty$ almost surely as $t \to \infty$, which is equivalent to saying that ruin is not a certain event. A sufficient condition to guarantee the latter, is the following net profit condition

$$\lim_{t\to\infty} \frac{E[U(t)]}{t} > 0. \tag{2.2.4}$$

For the proof of condition (2.2.4), we refer to Rolski et al. (1999). Therefore, the random variable τ may be defective; that is, it may happen that $P(\tau < \infty) < 1$.

Note that as we consider finite-time ruin probabilities, no profit condition has to be satisfied from a theoretical point of view in order to have $\psi(u, t) < 1$, for a fixed $t > 0$. In practice, it is more likely that the surplus is checked at regular intervals and $\psi(u, t)$ indicates that the company has to take action in order to make the business profitable.

To complete the characterization of the surplus process, in the following two subsections we describe the claim number and the claim size processes.

2.2.1. The claim number process

The claim number process $\{N(t), t \geq 0\}$ is a counting process, in the sense that the following conditions are satisfied:

i. $N(0) = 0$;

ii. $N(t)$ has non-negative integer values for all $t \geq 0$;

iii. If $0 \leq s \leq t$, then $N(s) \leq N(t)$.

For $0 \leq s < t$, $N(t) - N(s)$ represents the number of claims occurring in the time interval $(s, t]$. The counting processes may have the following properties.

Definition 2.2.6. *The counting process* $\{N(t), t \geq 0\}$ *is said to be with independent increments, if for all* $n \geq 1$ *and time points* $0 \leq t_0 < t_1 < ... < t_n$, *the increments* $N(t_0)$, $N(t_1) - N(t_0)$,...,$N(t_n) - N(t_{n-1})$ *are independent random variables (the number of events in disjoint intervals of time are independent).*

Definition 2.2.7. *The counting process* $\{N(t), t \geq 0\}$ *is said to possess stationary increments if for* $0 < s < t$ *the distribution of* $N(t) - N(s)$ *depends only on the length* $(t - s)$ *of the time interval* $[s, t]$ *and not on the values of* s *and* t.

Stationary and independent increments imply that the process can be thought of intuitively as starting over at any point in time.

Counting processes used in claim number modeling are for example, Poisson and renewal processes, which are presented below.

2.2.1.1. *Poisson process*

Due to its simplicity, the most popular choice for the claim arrivals process is the Poisson process, which is defined as follows.

Definition 2.2.8. *A counting process* $\{N(t), t \geq 0\}$ *is an homogeneous Poisson process with intensity rate* $\lambda > 0$ *if the following conditions hold:*

1. $N(0) = 0$;

2. The process has independent increments;

3. For $s < t$, *the number of claims in the interval* $(s, t]$ *has a Poisson distribution with mean* $\lambda(t - s)$:

$$Pr\left(N(t) - N(s) = n\right) = \frac{\left[\lambda(t-s)\right]^n e^{-\lambda(t-s)}}{n!}, \quad n = 0, 1, ...$$

Condition 3 implies that the increments are stationary because the distribution of $N(t) - N(s)$ depends only on the length $(t - s)$ of the time interval $[s, t]$, and hence, the expected number of events in the interval $[0, t]$ is $E[N(t)] = \lambda t$.

A characterization of the inter-arrival times is given by the following result.

Proposition 2.2.1. *The claim inter-arrival times $\{T_n\}_{n\geq 1}$ are independent and identically distributed, each having an exponential distribution with mean $1/\lambda$.*

The proof of this proposition can be found in Klugman et al. (2004).

A consequence of Proposition 2.2.1, due to the lack of memory property (discussed in Proposition 1.4.3) of the exponentially distributed claim inter-arrival times, is presented as follows.

Proposition 2.2.2. *From a fixed point in time $t > 0$, the time until the next claim occurs is also exponentially distributed with mean $1/\lambda$.*

Proof. Assume that s is the time of the last claim prior to time t, letting $s = 0$ if no claim occurs prior t. Now define A_s and A_t to be the time until the next claim from times s and t, respectively. Since A_s is exponentially distributed with parameter λ, then $P(A_t > x) = P(A_s > t - s + x \mid A_s > t - s) = e^{-\lambda x}$, which completes the proof. $\square$

Combining Proposition 2.2.1 with Lemma 1.1.1 yields the following result.

Proposition 2.2.3. *The n-th claim arrival time, $\sigma_n = T_1 + ... + T_n$, follows an Erlang distribution with parameters n and λ, that is, the p.d.f. of σ_n is*

$$f(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0. \end{cases}$$

Other properties of the Poisson process are listed below.

Proposition 2.2.4. *Let $\{N(t), t \geq 0\}$ be a Poisson process. Then,*

(1) With probability one, $N(t)/t \to \lambda$ when $t \to \infty$. This means that the intensity measures the average frequency of claim arrivals.

(2) The probabilities of occurrence of events in a small interval of length h are given as follows:

$$P(N(t+h) - N(t) = 0) = e^{-\lambda h} = 1 - \lambda h + o(h),$$
$$P(N(t+h) - N(t) = 1) = \lambda h e^{-\lambda h} = \lambda h + o(h),$$

$$P(N(t+h)-N(t)\geq 2)=o(h).$$

(3) The probability of occurrence of an infinite number of events in $[0,t]$ *is zero; that is, for any* $t>0$, $P(N(t)<\infty)=1$.

(4) The probability generating function (p.g.f.) is computed as

$$P_{N(t)}(z)=\sum_{n=0}^{\infty}z^nP(N(t)=n)=e^{\lambda t(z-1)}.$$

(5) The sum of independent Poisson processes $N_1(t),\ldots,N_k(t)$ *with mean values* $\lambda_1 t,\ldots,\lambda_k t$, *respectively, is also a Poisson process with mean* $\sum_{i=1}^{k}\lambda_i t$.

For the proof of these properties, we refer to Kannan (1979).

Extensive discussions of the Poisson process, from both applied and theoretical views can be found for example, in Cramér (1930), Bühlmann (1970), Çinlar (1975), Gerber (1979).

2.2.1.2. *Renewal process*

A renewal process is a more general case of the Poisson process in the sense that the inter-arrival times of the process do not necessarily follow the exponential distribution.

Definition 2.2.9. *A counting process* $N(t)$ *is called an ordinary renewal process if the inter-arrival times* $\{T_n\}_{n\geq 1}$ *form a sequence of positive independent and identically distributed random variables.*

Similar to the case of the Poisson process, we have the following property here.

Proposition 2.2.5. *The renewal process* $N(t)$ *has finite values for each* t:

$$P(N(t)<\infty)=1 \text{ for any } t>0.$$

For the proof, we refer to Kannan (1979).

According to relation (2.2.1), the probability that exactly n events occur by time t can be written as

$$P(N(t)=n)=P(\sigma_n\leq t)-P(\sigma_{n+1}\leq t)=F_n(t)-F_{n+1}(t),$$

where $F_n(t)$ is the cumulative distribution function for the n-th claim arrival time.

Definition 2.2.10. *The mean value function of the renewal process, denoted by* $m(t)$, *is the expected number of arrivals up to time* t: $m(t)=E[N(t)]$.

Proposition 2.2.6. *The renewal function $m(t)$ is equal to the sum of the distribution functions for all claim arrival times.*

PROOF. $$m(t) = E[N(t)] = \sum_{n=1}^{\infty} nP(N(t) = n) = \sum_{n=1}^{\infty} n[F_n(t) - F_{n+1}(t)]$$

$$= F_1(t) + \sum_{n=1}^{\infty} (n+1)F_{n+1}(t) - \sum_{n=1}^{\infty} nF_{n+1}(t) = \sum_{n=1}^{\infty} F_n(t),$$

which establishes the result. □

Exact formulas for $m(t)$ can be complicated. To avoid this difficulty, we turn our attention to the asymptotic behavior of the number of renewals as time tends to infinity, which is illustrated by the following proposition.

Proposition 2.2.7. *Let $\lambda^{-1} = E[T_n]$ be the mean inter-arrival time and $m(t)$ be the expected number of renewals by time t. Then,*

(1) With probability one, $\frac{N(t)}{t} \to \lambda$ as $t \to \infty$.

(2) $\frac{m(t)}{t} \to \lambda$ as $t \to \infty$.

For the proof, we refer to Durrett (1999).

Renewal processes do not, in general, have stationary or independent increments. The Poisson process is the only renewal process that has independent and stationary increments. For more details on these properties, we refer to Nelson (1995).

2.2.2. The claim size process

For the purpose of modeling a risk process, the claim size distribution is just as important as the claim number process.

This subsection presents briefly some of the most popular classes of distributions which have been used to model the claims $\{X_n\}_{n\geq 1}$. These distributions are roughly classified into two groups, light-tailed distributions and heavy-tailed distributions. The tail of a distribution or, more properly, the right tail is that part that reveals probabilities about large values. It is of interest to actuaries because it is the occurrence (or lack) of large claims that is most influential on profits.

Random variables that tend to assign higher probabilities to large values are said to be heavy-tailed. A formal definition of these types of distributions follows.

Definition 2.2.11. *The class of light-tailed distributions consists of those distributions F with a moment generating function that satisfies*

$$M(r) = \int_0^\infty e^{rx} dF(x) < \infty \quad \text{for some} \quad r > 0.$$

In contrast, F is heavy-tailed if

$$M(r) = \int_0^\infty e^{rx} dF(x) = \infty \quad \text{for all} \quad r > 0.$$

As examples of light-tailed distributions we mention exponential, gamma, or hyperexponential (defined as a finite mixture of exponential distributions); for further details, we refer to Asmussen and Albrecher (2010).

For classes of heavy-tailed distributions, different more restrictive definitions are often used, as we will see in the following definition.

Definition 2.2.12. *Let F be a distribution function on $[0,\infty)$ such that $\overline{F}(x) = 1 - F(x) > 0$ for all $x \geq 0$. Then*

1. F is said to belong to the consistent variation class $\mathcal{C}$ if

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1 \text{ or, equivalently, } \lim_{y\downarrow 1} \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

2. F is said to belong to the dominant variation class $\mathcal{D}$ if the relation

$$\limsup_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} < \infty$$

holds for some (or, equivalently, for all) $0 < t < 1$.

3. F is said to belong to the subexponential class $\mathcal{S}(n)$ if

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \tag{2.2.5}$$

holds for all $n = 2, 3, ...$ (or, equivalently, for $n = 2$).

4. F is said to belong to the long-tailed class $\mathcal{L}$ if the relation

$$\lim_{x\to\infty} \frac{\overline{F}(x+t)}{\overline{F}(x)} = 1 \tag{2.2.6}$$

holds for all real t.

In (2.2.5), F^{*n} denotes the n-fold convolution of F with itself. It can be obtained as $F^0(x) = 1$ if $x \geq 0$ and 0 otherwise, and

$$F^{*n}(x) = \int_{-\infty}^{\infty} F^{*(n-1)}(x-y)dF(y) \quad \text{for } n \geq 1. \tag{2.2.7}$$

The property (2.2.5) shows that the only way $X_1+...+X_n$ can get large is roughly by one of the X_i becoming large, whereas the property (2.2.6) means that for all t, given that $X > x$ for a large level x, then also $X > x+t$ with a large probability.

Throughout the thesis we adopt the notation $\mathcal{S}$ for the subexponential class.

These heavy-tailed classes satisfy the following inclusions.

Proposition 2.2.8. $\mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$.

For the proof, we refer to Embrechts et al. (1997).

Some of the properties characterizing distributions that belong to the subexponential class are given by the following lemma.

Lemma 2.2.1. *Let $F \in \mathcal{S}$. Then*

i. For every $\epsilon > 0$, $e^{\epsilon x}\overline{F}(x) \to \infty$ as $x \to \infty$;

ii. For every $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that the inequality

$$\overline{F^{*n}}(x) \leq C(\epsilon)(1+\epsilon)^n \overline{F}(x) \tag{2.2.8}$$

holds for all $n = 1, 2, ...$ and all $x \geq 0$;

iii. If X_1, X_2,...are independent and identically distributed random variables with common distribution function $F \in \mathcal{S}$ then, for every $n = 2, 3, ...$,

$$P(X_1 + X_2 + ... + X_n > x) \sim P(\max\{X_1, X_2, ..., X_n\} > x) \sim n\overline{F}(x). \tag{2.2.9}$$

For the proof, we refer to Embrechts et al. (1997).

Distributions like Pareto, lognormal, Weibull belong to the class $\mathcal{S}$. For details on these examples, we refer to Asmussen and Albrecher (2010).

Property (2.2.9) explains why $\mathcal{S}$ can be used to model large claim amounts, in the sense, that asymptotically, the accumulation of n successive claims is governed by one very big claim. This is indeed often observed in actuarial applications: the aggregate loss is determined by the largest loss. For this reason, the subexponential class is an important class of heavy-tailed distributions.

2.3. Univariate ruin models

Ruin theory for the univariate risk process described by Definition 2.2.1 has been extensively discussed in the literature and we refer here to some of the books which deal more extensively with this topic, for example, Beard et al. (1984), Grandell (1991), Panjer and Willmot (1992), Bowers et al. (1997), Rolski et al. (1999), Asmussen (2000), Klugman et al. (2004), Dickson (2005), or Asmussen and Albrecher (2010).

In the first two subsections, we describe two models that have been extensively analyzed in the actuarial literature, namely the classical risk model (or Cramér-Lundberg model) and its generalization, the renewal model (or Sparre Andersen model), for the purpose of modeling the surplus process from Definition 2.2.1. These two models involve Poisson and renewal processes, which have been introduced in Subsections 2.2.1.1 and 2.2.1.2, respectively.

A review of the cases where it is possible to find closed form expressions for the ruin probabilities $\psi(u)$ and $\psi(u,t)$, for a fixed $t > 0$, associated to these two models, can be found in Asmussen and Albrecher (2010).

Our goal is to analyze ruin probabilities associated to the multivariate risk processes. Since deriving explicit and closed form expressions for the ruin probabilities is rather difficult, we turn our attention to obtaining bounds and studying the asymptotic behavior. Therefore, Subsections 2.3.3 to 2.3.5 contain results regarding bounds and approximations of the ruin probabilities in the classical and renewal models. This review will establish the necessary background for Chapters 4 and 5, where we consider multivariate versions of these models.

2.3.1. Classical risk model

A sound mathematical basis for the stochastic modeling of insurance risk can be traced back to the pioneering work of Filip Lundberg (1926, 1932) and Harald Cramér (1930, 1955) who proposed the following model.

Definition 2.3.1. *The classical Cramér-Lundberg risk model associated to the surplus process from Definition* 2.2.1 *assumes that the claim arrivals process* $N(t)$

is an homogeneous Poisson process with intensity $\lambda > 0$*, which is independent of the claim sizes* $\{X_n\}_{n\geq 1}$.

Proposition 2.2.1 implies that in the classical risk model the inter-arrival times $\{T_n\}_{n\geq 1}$ are independent and exponentially distributed, each having mean $1/\lambda$.

Also, in this model, the aggregate claims process $S(t) = \sum_{k=1}^{N(t)} X_k$ is a compound Poisson process with mean $E[S(t)] = \lambda\mu t$ and thus, the net profit condition (2.2.4) becomes

$$\lim_{t\to\infty} \frac{E[U(t)]}{t} = c - \lim_{t\to\infty} \frac{\lambda\mu t}{t} = c - \lambda\mu > 0. \tag{2.3.1}$$

Let $c = (1+\theta)\lambda\mu$, where $\theta > 0$ is called the safety loading coefficient. The condition $\theta > 0$ can be interpreted as if premiums received per unit time exceed the expected claim payments per unit time and thus, it can be considered an index to measure the safety (non-ruin) of an insurance company.

The following result describes the increments of the aggregate claims process.

Proposition 2.3.1. *If* $\{N(t), t \geq 0\}$ *is an homogeneous Poisson process, then the compound Poisson process* $S(t) = \sum_{k=1}^{N(t)} X_k$*,* $t \geq 0$ *has independent and stationary increments.*

For the proof, we refer to Kannan (1979).

2.3.1.1. *The Adjustment Coefficient*

Let X be a generic random variable for the claim amounts $\{X_n\}_{n\geq 1}$. Assume that the moment generating function (m.g.f.) of X, defined as

$$M_X(r) = E[e^{rX}] = \int_0^\infty e^{rx} dF(x),$$

exists and there exists some quantity r_0, $0 < r_0 \leq \infty$, such that $M_X(r)$ is finite for all $r < r_0$ with $\lim_{r\uparrow r_0} M_X(r) = \infty$.

In this context, we have the following definition.

Definition 2.3.2. *The adjustment coefficient (or Lundberg exponent), denoted by* R*, is defined to be the unique positive root of the equation*

$$\lambda M_X(r) - \lambda - cr = 0, \tag{2.3.2}$$

so that R *is given by*

$$\lambda M_X(R) = \lambda + cR. \tag{2.3.3}$$

Condition (2.3.3) is also known as the Cramér-Lundberg condition and is sometimes called a small claim condition. The existence and singularity of a strict positive solution to equation (2.3.2) is established by the fact that the function

$$g(r) = \lambda M_X(r) - \lambda - cr \tag{2.3.4}$$

is a decreasing function at zero, convex and continuous on $[0, r_0)$, satisfying $g(0) = 0$ and $\lim_{r \uparrow r_0} g(r) = \infty$ [for a proof, see, for example, Dickson (2005)].

Therefore, as we saw in Subsection 2.2.2, claim size distributions ("small claims") that allow for the construction of the adjustment coefficient are for example, exponential, gamma, whereas for distributions ("large claims") like log-normal, Pareto, this coefficient does not exist.

Example 2.3.1. *If X is exponentially distributed with parameter α and hence, having mean $\mu = 1/\alpha$, then the moment generating function is*

$$M_X(t) = \frac{\alpha}{\alpha - t}, \quad t < \alpha, \tag{2.3.5}$$

and the positive solution of equation (2.3.2) *is $R = \theta\alpha/(1+\theta)$.*

Remark 2.3.1. *According to Grandell (1991), the case where there exists $0 < r_0 < \infty$ such that $M_X(r) < \infty$ for all $r \leq r_0$ and $M_X(r) = \infty$ for $r > r_0$ is excluded, while Asmussen and Albrecher (2010) illustrated this case by considering the inverse Gaussian distribution and showed that R exists provided $M(r_0) \geq 1 + r_0/\lambda$. As pointed out by Asmussen and Albrecher (2010), this case is a somewhat special situation.*

The following proposition gives an expression for the ruin probability in terms of the adjustment coefficient and its proof can be found in Grandell (1991).

Proposition 2.3.2. *If $R > 0$ is the adjustment coefficient defined by* (2.3.3)*, then*

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(\tau)} | \tau < \infty]}. \tag{2.3.6}$$

If the claim sizes $\{X_n\}_{n \geq 1}$ are exponentially distributed, the denominator in (2.3.6) can be computed and yields the following result for the ruin probability.

Proposition 2.3.3. *If X is exponentially distributed with mean $\mu > 0$, then*

$$\psi(u) = \frac{1}{1+\theta} \exp\left[-\frac{\theta u}{\mu(1+\theta)}\right]. \tag{2.3.7}$$

For the proof, we refer to Cramér (1955).

In the general (non-exponential) case, the denominator in (2.3.6) is not easily evaluated, but gives an estimate, known as Lundberg's inequality, for the ruin probability, as we will see in Subsection 2.3.3.

Using the notion of equilibrium distribution, an equivalent expression for condition (2.3.3) can be obtained as follows.

Definition 2.3.3. *The equilibrium distribution of a distribution function F is defined by*

$$G(x) = \frac{1}{\int_0^\infty [1-F(y)]dy} \int_0^x [1 - F(y)]dy.$$

Then, the Cramér-Lundberg condition (2.3.3) is equivalent to

$$\int_0^\infty e^{Rx} dG(x) = 1 + \theta. \tag{2.3.8}$$

The ruin probability $\psi(u)$ can be expressed in terms of the equilibrium distribution of F as follows.

Proposition 2.3.4. *The ruin probability associated to the classical risk model is given by*

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta} \right)^n [1 - G^{*n}(u)], \;\; u \geq 0,$$

where G^{*n} is the n-fold convolution distribution function of G, defined by (2.2.7).

The formula from Proposition 2.3.4 is known as Beekman's convolution series in actuarial science [Beekman, (1968)]. A disadvantage of this formula is that it involves an infinite number of convolutions, which can be difficult to compute in practice.

In general, it is difficult to derive explicit and closed form expressions for the ruin probability. However, under suitable conditions, some approximations to the ruin probability can be found, as will be presented in Subsection 2.3.3.

2.3.2. Renewal risk model

In the actuarial literature, the renewal risk model is referred to as the Sparre Andersen model, after E. Sparre Andersen who proposed a generalization of the classical risk model in a paper to the 1957 International Congress of Actuaries in New York [Andersen, (1957)]. This model is defined as follows.

Definition 2.3.4. *The Sparre Andersen risk model associated to the surplus process from Definition* 2.2.1 *assumes that the claim arrivals process* $N(t)$ *is an ordinary renewal process, which is independent of the claim sizes* $\{X_n\}_{n\geq 1}$.

Following Definition 2.2.9, the claim inter-arrival times $\{T_n\}_{n\geq 1}$ are positive i.i.d. random variables and let us assume that their common distribution function is $Q(x) = P(T_n \leq x)$ and finite mean value is

$$E[T_n] = \int_0^\infty [1 - Q(x)]dx = \lambda^{-1} > 0. \tag{2.3.9}$$

Thus, the Cramér-Lundberg model corresponds to the particular case where the claim inter-arrival times are exponentially distributed.

The aggregate claims process $S(t) = \sum_{k=1}^{N(t)} X_k$, also called renewal reward process, has the mean value $E[S(t)] = \mu m(t)$, and hence, the net profit condition (2.2.4) becomes

$$\lim_{t\to\infty} \frac{E[U(t)]}{t} = c - \lim_{t\to\infty} \frac{\mu m(t)}{t} = c - \lambda\mu > 0, \tag{2.3.10}$$

in view of assumption (2.3.9) and Proposition 2.2.7. As in the classical risk model, we let $c = (1+\theta)\lambda\mu$, where $\theta > 0$ is the safety loading coefficient.

2.3.2.1. *The Adjustment Coefficient*

Let X and T be generic random variables of the claim sizes $\{X_n\}_{n\geq 1}$ and of the claim inter-arrival times $\{T_n\}_{n\geq 1}$, respectively.

Under the assumption that the moment generating function of X exists and there exists some quantity r_0, $0 < r_0 \leq \infty$, such that $M_X(r)$ is finite for all $r < r_0$ with $\lim_{r\uparrow r_0} M_X(r) = \infty$, we have the following definition.

Definition 2.3.5. *The adjustment coefficient (or Lundberg exponent), denoted by* R*, is defined to be the unique positive root of the equation*

$$E[e^{-crT}] \times E[e^{rX}] - 1 = 0, \tag{2.3.11}$$

so that R is given by

$$E[e^{-cRT}] \times M_X(R) = 1. \tag{2.3.12}$$

Let us denote $h(r) = E[e^{-crT}] \times M_X(r) - 1$. As was shown in Grandell (1991), the function $h(r)$ is a decreasing function at zero, convex, and continuous on $[0, r_0)$, satisfying $h(0) = 0$ and $\lim_{r \uparrow r_0} h(r) = \infty$. All of these properties of $h(r)$ establish the existence and singularity of a strict positive solution for equation (2.3.11).

From relation (2.2.2), it follows that the ruin probability can be expressed as $\psi(u) = P(N_u < \infty)$, where N_u is the number of claims causing ruin, that is, $N_u = \min\{n \geq 1 : Z_n > u | U(0) = u\}$ with Z_n being defined by (2.2.3). In this framework, an expression for the ruin probability follows.

Proposition 2.3.5. *If R is the adjustment coefficient defined by* (2.3.12)*, then*

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-R(u - Z_{N_u})} | N_u < \infty]}. \tag{2.3.13}$$

For the proof, we refer to Grandell (1991).

If the claim sizes $\{X_n\}_{n \geq 1}$ are exponentially distributed, then the following result for formula (2.3.13) is obtained.

Proposition 2.3.6. *If X is exponentially distributed with mean value $\mu > 0$, then*

$$\psi(u) = (1 - \mu R) e^{-Ru}.$$

For the proof, we refer to Grandell (1991).

The adjustment coefficient R is used in the derivation of approximations and bounds for the probability of ruin, as we will see in the next section.

Since the renewal risk model is more flexible than the classical risk model, much research on the former has been carried out in recent years. For details on the Sparre Andersen risk model in general such as calculation of ruin probability, see for example, Grandell (1991), Rolski et al. (1999), Asmussen and Albrecher (2010), Willmot and Lin (2001), and references therein.

2.3.3. Review on bounds and asymptotic behavior

Since the ruin probability is intractable in most cases, finding computable bounds for $\psi(u)$ is of significant importance in reliability modeling and risk management [see, for example, Asmussen (2000)]. If ruin occurs, it is interpreted as if the company has to take action in order to make the business profitable.

The results regarding the bounds and the behavior of $\psi(u)$ as $u \to \infty$, range from the exponential type estimates based on Cramér-Lundberg conditions of type (2.3.3) (associated to the classical risk model) or (2.3.12) (associated to the renewal risk model) to the estimates of $\psi(u)$ when these conditions are not valid.

2.3.3.1. *Light-tailed claim size distributions*

We start by considering the case of claim sizes modeled by light-tailed distributions, and hence, conditions (2.3.3) and (2.3.12) are satisfied.

The pioneering works on approximations to the ruin probability were achieved by Cramér and Lundberg in 1930, under the Cramér-Lundberg condition given by (2.3.3), or equivalently, by (2.3.8). In this context, the Cramér-Lundberg asymptotic formula is given by the following proposition.

Proposition 2.3.7. *Let R be the adjustment coefficient defined by condition* (2.3.3) *and G be the equilibrium distribution of F (see Definition* 2.3.3*).*

(1) If $\int_0^\infty xe^{Rx}dG(x) < \infty$, then

$$\psi(u) \sim \frac{\theta\mu}{R\int_0^\infty ye^{Ry}[1-F(y)]dy} e^{-Ru} \ \text{as } u \to \infty. \tag{2.3.14}$$

(2) If $\int_0^\infty xe^{Rx}dG(x) = \infty$, then $\psi(u) = o(e^{-Ru})$ as $u \to \infty$.

The most famous of the bounds (especially the upper bound) of the ruin probability is attributed to Lundberg (1932).

Proposition 2.3.8. *(Lundberg's inequality) If R is the adjustment coefficient defined by condition* (2.3.3)*, then*

$$\psi(u) \le e^{-Ru}, \ u \ge 0. \tag{2.3.15}$$

Proof. The proof of this inequality follows immediately from Proposition 2.3.2, using the fact that the denominator in (2.3.6) is greater than one since $U(\tau) < 0$ on $(\tau < \infty)$ and $R > 0$. $\square$

The asymptotic formula (2.3.14) provides an exponential asymptotic estimate for the ruin probability as $u \to \infty$, while Lundberg's inequality (2.3.15) gives an exponential upper bound for the ruin probability for all $u \geq 0$. These two results constitute the well-known Cramér-Lundberg approximations for the ruin probability in the classical risk model and they have become two standard results on ruin probabilities in risk theory.

The original proofs of the Cramér-Lundberg approximations were based on Wiener-Hopf methods and can be found in Cramér (1930, 1955) and Lundberg (1926, 1932). However, these two results can be proved in different ways nowadays. For example, the martingale approach of Gerber (1973, 1979), Wald's identity in Ross (1996), and the induction method in Goovaerts et al. (1990) have been used to prove the Lundberg's inequality. The asymptotic formula (2.3.14) can be obtained from the renewal theorem regarding the solution of a defective renewal equation, see, for example, Feller (1971).

In Subsection 2.3.4 we present the martingale technique in order to derive the Lundberg' inequality.

The Cramér-Lundberg asymptotic formula (2.3.14) is exact when the claim sizes are exponentially distributed, that is, it becomes (2.3.7) . Under the Cramér-Lundberg condition (2.3.3), the following result can be proved [e.g. Cai and Garrido (1999a), Willmot and Lin (2001)].

Proposition 2.3.9. *The ruin probability associated to the classical risk model satisfies*

$$\psi(u) \leq \beta e^{-Ru}, \; u \geq 0, \tag{2.3.16}$$

where $0 < \beta \leq 1$ *is a constant given by* $\beta^{-1} = \inf_{0 \leq t < \infty} \frac{1}{e^{Rt}[1-G(t)]} \int_t^{\infty} e^{Ry} dG(y)$.

This improved Lundberg upper bound (2.3.16) equals the ruin probability when the claim sizes are exponentially distributed. In fact, the constant β in (2.3.16) has an explicit expression of $\beta = 1/(1+\theta)$ if the distribution F has a decreasing failure rate; see, for example, Willmot and Lin (2001) for details.

The Cramér-Lundberg approximation is also available for ruin probabilities in the renewal risk model presented in Subsection 2.3.2.

The Cramér-Lundberg approximation is given by the following proposition.

Proposition 2.3.10. *If the claims occur according to a renewal process and R is the adjustment coefficient defined by condition* (2.3.12)*, then*

$$\psi(u) \sim Ce^{-Ru}, \ (0 < C < \infty), \ \textit{as } u \to \infty. \tag{2.3.17}$$

For the proof, we refer to Grandell (1991).

Note that the constant C in (2.3.17) cannot in general be calculated explicitly, as was done in the classical risk model given by relation (2.3.14). Thorin (1974) has given C in a form which involves certain auxiliary functions. That form can probably be used in numerical solutions.

Also, Lundberg's inequality in the renewal risk model follows.

Proposition 2.3.11. *(Lundberg's inequality) If R is the adjustment coefficient defined by* (2.3.12)*, then*

$$\psi(u) \le e^{-Ru}. \tag{2.3.18}$$

Proof. The proof of this inequality follows immediately from Proposition 2.3.5, using the fact that the denominator in (2.3.13) is greater than one since $u - Z_{N_u} < 0$ on $(N_u < \infty)$ and $R > 0$. $\square$

Lundberg's inequality in the ordinary renewal case was first proved by Sparre Andersen (1957), by completely different methods.

In what follows, a brief review of approximations to ruin probabilities when the claim size distribution is heavy-tailed is presented.

2.3.3.2. *Heavy-tailed claim size distributions*

When the moment generating function of a distribution does not exist or a distribution is heavy-tailed such as Pareto and lognormal distributions, the Cramér-Lundberg conditions (2.3.3) or (2.3.12) are not valid. In these cases, an exponential upper bound for $\psi(u)$ may not exist and the asymptotic behavior of the ruin probabilities is totally different from when the aforementioned conditions hold.

For example, DeVylder and Goovaerts (1984) proved that for a subexponential claim size distribution F, no exponential upper bound exists for $\psi(u)$.

In view of expression (2.2.2), we have that the survival probability is given by

$$\phi(u) = P\left(\sup_{n\geq 0} Z_n \leq u\right),$$

where $\{Z_n\}_{n\geq 0}$ are defined by relation (2.2.3). Thus, the survival probability represents the distribution function of the ultimate maximum of $\{Z_n\}_{n\geq 0}$.

The following result shows that ruin is asymptotically determined by a large claim when considering the class of subexponential distributions.

Proposition 2.3.12. *Consider the Cramér-Lundberg model with the net profit condition $c - \lambda\mu > 0$. Then the following statements are equivalent:*
(1) $G \in \mathcal{S}$,
(2) $\phi \in \mathcal{S}$,
(3) $\psi(u) = 1 - \phi(u) \sim \frac{\lambda}{c-\lambda\mu}\int\limits_u^\infty [1 - F(y)]dy \quad as \quad u \to \infty$,
where G is the equilibrium distribution from Definition 2.3.3.

The proof of this proposition can be found in Teugels and Veraverbeke (1973), or in Embrechts and Veraverbeke (1982).

The question arises whether the condition $G \in \mathcal{S}$ can be replaced by the requirement that the right tail of the claim size distribution F is subexponential. This is an open problem. However, with some important practical distributions, such as Pareto or lognormal, one can indeed prove so [for detailed discussions, see Embrechts and Veraverbeke (1982)].

An asymptotic estimate of the ruin probability when the distribution F belongs to the dominant variation class $\mathcal{D}$ (see Definition 2.2.12) is presented below.

Proposition 2.3.13. *Consider the Cramér-Lundberg model with the net profit condition $c - \lambda\mu > 0$. If $F \in \mathcal{D}$, then $\psi(u) \sim \frac{\lambda}{c-\lambda\mu}\int\limits_u^\infty [1 - F(y)]dy$ as $u \to \infty$.*

The proof can be found in Embrechts et al. (1997).

Now for the renewal risk model we have the following asymptotic estimate for the ruin probability.

Proposition 2.3.14. *Assume the Sparre Andersen model with the net profit condition $c - \lambda\mu > 0$. If $G \in \mathcal{S}$, then*
(1) $\phi \in \mathcal{S}$,
(2)

$$\psi(u) = 1 - \phi(u) \sim \frac{\lambda}{c - \lambda\mu} \int_u^\infty [1 - F(y)]dy \quad as \quad u \to \infty. \tag{2.3.19}$$

For proof, we refer to Embrechts and Veraverbeke (1982).

Comparing this result to the corresponding result given by Proposition 2.3.12 in the classical risk model, it should be noted that in the Poisson case, the class $\mathcal{S}$ is really the class for which all one sided implications are valid in both ways. Formula (2.3.19) shows that ruin depends on the distribution of claims F only through its behavior in the tail. Since the derivation of this celebrated formula (2.3.19) for the infinite time ruin probability $\psi(u)$, there has been great interest in the study of asymptotic behavior of ruin probabilities for heavy-tailed claims.

Also, in the context of heavy-tailed distributions, the following propositions give asymptotic estimates of the finite-time ruin probability $\psi(u,t)$ for fixed t in both classical and renewal risk models.

Proposition 2.3.15. *Under the Cramér-Lundberg model, if the claims are independent and subexponentially distributed, then*

$$\psi(u,t) \sim \lambda t \overline{F}(u), \;\; u \to \infty. \tag{2.3.20}$$

For the proof, we refer to Asmussen (2000). A more general result was obtained by Kaas and Tang (2003).

For the renewal case, we first need to introduce the following definition, according to Tang and Tsitsiashvili (2003).

Definition 2.3.6. *Define*

$$\overline{F}_\star(y) = \liminf_{x\to\infty} \tfrac{\overline{F}(xy)}{\overline{F}(x)}, \;\; J_F^+ = -\lim_{y\to\infty} \tfrac{\log \overline{F}_\star(y)}{\log y},$$

and call J_F^+ the upper Matuszewska index of the distribution F.

Tang (2004) proved the following result regarding the asymptotic behavior of the finite-time ruin probability in the renewal case for the class of claim distributions having consistent variation (see Definition 2.2.12).

Proposition 2.3.16. *Under the Sparre-Andersen model, if* $F \in \mathcal{C}$, $E[T_1^p] < \infty$ *for some* $p > J_F^+ + 1$, *and assuming the net profit condition* $c - \lambda\mu > 0$, *then the relation*

$$\psi(u,t) \sim \frac{\lambda}{c-\lambda\mu} \int\limits_{u}^{u+\frac{c-\lambda\mu}{\lambda}m(t)} [1-F(y)]dy \qquad (2.3.21)$$

holds uniformly for all $t \in \Lambda = \{t : m(t) > 0\}$ *as* $u \to \infty$, *that is,*

$$\lim_{u\to\infty} \sup_{t\in\Lambda} \left| \frac{\psi(u,t)}{\frac{\lambda}{c-\lambda\mu} \int\limits_{u}^{u+\frac{c-\lambda\mu}{\lambda}m(t)} [1-F(y)]dy} - 1 \right| = 0.$$

Leipus and Siaulys (2007) established the asymptotic result (2.3.21) under mild additional assumptions on the subexponential distribution of the claim size X and on the distribution of the inter-arrival time T.

2.3.3.3. *General claim size distributions*

We conclude this review by presenting bounds of the ruin probabilities when the claim sizes can be modeled by any positive distributions, under the Cramér-Lundberg model. More specifically, a truncated Lundberg condition that applies to any positive claim size distribution with a finite mean was proposed by Dickson (1994) by assuming that for any $u > 0$, there exists a constant $R_u > 0$ so that

$$\int\limits_{0}^{u} e^{xR_u} dG(x) = 1 + \theta. \qquad (2.3.22)$$

Under the truncated condition (2.3.22), Dickson (1994) derived an upper bound for any $0 \le u \le t$:

$$\psi(u) \le e^{-uR_t} + \frac{1-G(t)}{\theta + 1 - G(t)}. \qquad (2.3.23)$$

In this context, Cai and Garrido (1999b) gave a tighter upper bound than that of Dickson (1994) in (2.3.23) and a lower bound for the ruin probability, which are given by the following proposition.

Proposition 2.3.17. *For any $u > 0$, if there exists a constant R_u such that* (2.3.22) *holds, then*

$$\frac{\theta e^{-2uR_u} + 1 - G(u)}{\theta + 1 - G(u)} \leq \psi(u) \leq \frac{\theta e^{-uR_u} + 1 - G(u)}{\theta + 1 - G(u)}. \tag{2.3.24}$$

Even when the Cramér-Lundberg condition (2.3.3) holds, the upper bound in (2.3.24) may be tighter than the Lundberg upper bound (2.3.15); see Cai and Garrido (1999b) for details.

For more details on bounds and asymptotic estimates of the ruin probabilities $\psi(u)$ and $\psi(u, t)$ in the classical and renewal risk models, we refer to Embrechts and Veraverbeke (1982), Embrechts et al. (1997), Willlmot and Lin (2001), Tang (2004), or Asmussen and Albrecher (2010).

2.3.4. Bounds obtained using PDM processes and martingales

We consider now the ruin problem in the martingale framework. To the best of our knowledge, the first use of martingales in actuarial modeling is due to Gerber (1973, 1979) and DeVylder (1977).

Dassios and Embrechts (1989) proposed a martingale approach by reformulating the classical risk model in terms of a PDM process and derived suitable martingales via Proposition 2.1.5. In Chapters 4 and 5 of this thesis, we adopt this approach in order to derive Lundberg-type upper bounds for ruin probabilities in multivariate risk models.

The following three propositions illustrate this technique for obtaining Lundberg's inequality given by (2.3.15).

Let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t^U\}_{t\geq 0},\ P)$ be a filtered probability space which carries the surplus process $\{U(t), t \geq 0\}$ defined in Section 2.2. Here, $\mathcal{F}_t^U$ is the natural filtration generated by $U(t)$: $\mathcal{F}_t^U = \sigma\left(U(s) : 0 \leq s \leq t\right)$.

Following Proposition 2.1.1 and Proposition 2.3.1, we conclude that the surplus process $U(t)$ from the classical risk model, presented in Subsection 2.3.1, is a time-homogeneous Markov process with respect to the natural filtration generated by $U(t)$. Between the jumps, caused by the claims, the process is deterministic. Thus, $U(t)$ is a PDM process and its generator is given as follows.

Proposition 2.3.18. *The infinitesimal generator of the homogeneous Markov process $(U(t),t)$, acting on a function $f(z,t)$ belonging to its domain is given by*

$$\mathcal{A}f(z,t) = \frac{\partial}{\partial t}f(z,t) + c\frac{\partial}{\partial z}f(z,t) + \lambda\left[\int_0^\infty f(z-x,t)dF(x) - f(z,t)\right]. \quad (2.3.25)$$

Proof. In standard PDM process notation, the state space of the above process is $\mathbb{R}$, consider $\eta_t = 1$ so that $K = \{1\}$ meaning that in the non-ruin state, $U(t)$ takes values in $(0,\infty)$ and evolves as $U(t) = U(\sigma_i) + ct$ (σ_i being the time of the latest jump before t). As ξ_t it is defined the surplus process $U(t)$ and in between jumps $\xi_t = z + ct$ for some value z so that $\frac{d}{dt}(z+ct) = c\frac{\partial}{\partial z}(z+ct)$ thus, the differential operator is $\chi = c\frac{\partial}{\partial z}$. The transition measure is $p(z,dx) = dF(z-x)$ since at jump times σ_i, we have $U(\sigma_i) = U(\sigma_i-) - X$. The jump intensity is given by the hazard rate function of the exponentially distributed inter-arrival times, which is equal to the constant λ, as we saw in Section 1.3. Therefore, by Proposition 2.1.2, the generator of the process $(U(t),t)$ is expressed as (2.3.25). For $f(z,t)$ to belong to the domain of the generator $\mathcal{A}$, it is sufficient that $f(z,t)$ be differentiable with respect to z, t for all z, t and

$$\left| \int_0^\infty f(z-x,t)dF(x) - f(z,t) \right| < \infty,$$

which completes the proof. □

The next step is to obtain martingales needed in establishing inequality (2.3.15).

Proposition 2.3.19. *If $r \in [0, r_0)$ and $g(r)$ is given by* (2.3.4), *then the process*

$$Z(t) = e^{-tg(r)-rU(t)}, \ t \geq 0,$$

is a martingale with respect to the filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t^U\}_{t\geq 0},\ P)$.

Proof. According to Proposition 2.1.5, we have that for a function f belonging to the domain of the infinitesimal generator described by relation (2.3.25) such that $\mathcal{A}f = 0$, the process $\{f(U(t),t), t \geq 0\}$ is a martingale. As in Dassios and Embrechts (1989), we try a solution of the form $f(z,t) = \alpha(t)e^{-rz}$, differentiable with respect to z, where we can assume that $\alpha(0) = 1$. Then the equation $\mathcal{A}f = 0$ yields

$$\alpha'(t) + \alpha(t)g(r) = 0,$$

where $g(r)$ is given by (2.3.4), and the solution is $\alpha(t) = e^{-tg(r)}$. Therefore, the process $Z(t) = e^{-tg(r)-rU(t)}$ is a martingale. □

In what follows, using the martingale obtained in Proposition 2.3.19, we give the proof of Lundberg's inequality for the classical risk model.

Proof of Proposition 2.3.8*:*

Let τ be the ruin time from Definition 2.2.2. Choose $t_0 < \infty$ and consider $\tau \wedge t_0 = \min(\tau, t_0)$ which is a bounded stopping time. Thus, based on the martingale given by Proposition 2.3.19, it follows from Proposition 2.1.4 that

$$\begin{aligned} e^{-ru} &= E[Z(0)] = E[Z(\tau \wedge t_0)] \\ &= E[Z(\tau \wedge t_0)|\tau \leq t_0]P(\tau \leq t_0) + E[Z(\tau \wedge t_0)|\tau > t_0]P(\tau > t_0) \\ &\geq E[Z(\tau)|\tau \leq t_0]P(\tau \leq t_0), \end{aligned}$$

and since $U(\tau) < 0$ on $(\tau < \infty)$,

$$P(\tau \leq t_0) \leq \frac{e^{-ru}}{E[Z(\tau)|\tau \leq t_0]} \leq \frac{e^{-ru}}{E[e^{-\tau g(r)}|\tau \leq t_0]} \leq e^{-ru} \sup_{0 \leq t \leq t_0} e^{tg(r)}. \tag{2.3.26}$$

Let $t_0 \to \infty$ in (2.3.26). Then we obtain

$$\psi(u) \leq e^{-ru} \sup_{t \geq 0} e^{tg(r)}. \tag{2.3.27}$$

In order to get inequality (2.3.27) as tight as possible, r has to be chosen as large as possible under the restriction $\sup_{t \geq 0} e^{tg(r)} < \infty$. As we saw in Subsection 2.3.1, $g(r)$ is decreasing at zero, continuous and convex on $[0, r_0)$ and hence, the adjustment coefficient defined by (2.3.3) satisfies $R = \sup\{r | g(r) \leq 0\}$. Therefore, the inequality (2.3.15) is established. □

A similar derivation of this result, using martingale theory, is to be found in Chapter 9 of Gerber (1979).

In Chapter 5, we derive an upper bound for ruin probability in a multivariate renewal risk model using the martingale technique by formulating the risk model as a PDM process. Therefore, we are motivated to present this approach for the univariate renewal risk model, as was described by Dassios and Embrechts (1989).

In the Sparre Andersen model, the surplus process $U(t)$ is not a Markov process (unless $N(t)$ is a Poisson process, as it was discussed in Subsection 2.3.2) because the time since the last claim provides information on the time until the next claim occurs. However, it can be made Markovian by introducing a supplementary variable, namely, $V(t) = t - \sigma_{N(t)}$, which represents the time elapsed since the last claim before time t. Then the process $Y(t) = \{(U(t), V(t)), t \geq 0\}$ will be a Markov process with respect to the filtration $\mathcal{F}_t = \mathcal{F}_t^U \vee \mathcal{F}_t^V$. This technique, called backward Markovization technique, can be found in Cox (1955).

In Chapter 5, for the multivariate renewal process we will use this backward Markovization technique and a detailed proof of the Markov property will be given.

Remark 2.3.2. *Another way to Markovize the process would be to consider the process* $\{(U(t), W(t)), t \geq 0\}$*, where* $W(t) = \sigma_{N(t)+1} - t$ *denotes the time until the next claim and this approach is called forward Markovization technique. In this case, the filtration is* $\mathcal{F}_t = \mathcal{F}_t^U \vee \mathcal{F}_t^W$*: at any time it is known when the next claim will arrive. Note that this filtration is not observable.*

Using the standard notation from Definition 2.1.7, the countable component η_t is the constant 1, the uncountable one consists of $U(t)$, $V(t)$, and time t, and $M_1 = \mathbb{R} \times \mathbb{R}_+^2$. The process evolves deterministically as $u(t) = u_0 + ct$, $v(t) = v_0 + t$ until the time of the first claim σ_1. Then at the random jump σ_1, $V(\sigma_1) = 0$ and $U(\sigma_1) = U(\sigma_1 -) - X_1$, where X_1 is the size of the first claim, and so on. Therefore, according to Proposition 2.1.2, the infinitesimal generator of the process $(U(t), V(t), t)$ acting on an absolutely continuous function $f(z, v, t)$ has the form

$$\mathcal{A}f(z,v,t) = \frac{\partial}{\partial t} f(z,v,t) + c\frac{\partial}{\partial z} f(z,v,t) + \frac{\partial}{\partial v} f(z,v,t)$$

$$+\lambda(v)\left(\int_0^\infty f(z-y,0,t)dF(y) - f(z,v,t)\right), \tag{2.3.28}$$

where $\lambda(v)$ is the hazard rate of T; that is, $\lambda(v) = q(v)/(1 - Q(v))$, and for all t, z, $v > 0$, $E\,|f(z - X, 0, t) - f(z, v, t)| < \infty$.

Based on generator (2.3.28), Dassios and Embrechts (1989) further derived an exponential martingale via Proposition 2.1.5, and obtained results for the ruin probabilities. These results are given by the following two propositions.

Proposition 2.3.20. *If the net profit condition* (2.3.10) *holds, then for all* $\theta \geq 0$,

$$e^{-\theta t} e^{-r_\theta U(t)} \frac{M_X(r_\theta) e^{(\theta + c r_\theta) V(t)}}{1 - Q(V(t))} \int_{V(t)}^{\infty} e^{-(\theta + c r_\theta) x} q(x) dx \tag{2.3.29}$$

is a martingale, where r_θ *is the unique positive solution of*

$$M_X(r) \int_0^{\infty} e^{-(\theta + cr)x} q(x) dx = 1. \tag{2.3.30}$$

Following Proposition 2.3.20, it results that if $\theta = 0$, then r_0 is the unique positive solution of

$$M_X(r) \int_0^{\infty} e^{-crx} q(x) dx = 1, \text{ and}$$

$$e^{-r_0(U(t) - cV(t))} \times \frac{M_X(r_0) \int_{V(t)}^{\infty} e^{-c r_0 x} q(x) dx}{1 - Q(V(t))} \tag{2.3.31}$$

is a martingale. Based on martingale (2.3.31), Dassios and Embrechts (1989) derived an expression for the ruin probability as follows.

Proposition 2.3.21. *If* $V(0) = v_0 \geq 0$, *then the ruin probability is expressed as*

$$\psi(u) = \frac{e^{-r_0 u}}{E[e^{-r_0 U(\tau)} | \tau < \infty]} \times \frac{M_X(r_0) e^{c r_0 v_0} \int_{v_0}^{\infty} e^{-c r_0 x} q(x) dx}{1 - Q(v_0)}.$$

In Chapter 4, we extend a multivariate risk process by adding a diffusion process and this study motivates us to present the univariate classical risk model perturbed by diffusion in the following subsection.

2.3.5. Classical risk model perturbed by diffusion

The classical risk model perturbed by a diffusion was introduced by Gerber (1970) and in this framework a variety of ruin problems have been analyzed by many authors, for example, Dufresne and Gerber (1991), Veraverbeke (1993), Furrer and Schmidli (1994), Li and Garrido (2005), and a review of these models can be found in Schmidli (1999).

The perturbed risk model is obtained from the classical compound Poisson risk model by adding a diffusion process, that is, the surplus at time t will be of the form

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k + \sigma B(t),\ t \geq 0, \tag{2.3.32}$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the rate at which the premiums are received and the aggregate claims process $S(t) = \sum_{k=1}^{N(t)} X_k$ is a compound Poisson process. Here, $\{B(t), t \geq 0\}$ is a standard Brownian motion starting from zero, that is, $B(0) = 0$, the process $\{B(t), t \geq 0\}$ has independent and stationary increments and for every $t > 0$, $B(t)$ is normally distributed with mean zero and variance t and $\sigma > 0$ is the diffusion volatility coefficient. In addition, $\{X_k, k \geq 1\}$, $\{N(t), t \geq 0\}$ and $\{B(t), t \geq 0\}$ are assumed to be mutually independent.

The diffusion term in (2.3.32) expresses an additional uncertainty of the aggregate claims or of the premium income.

The ruin probability associated to the process $\{U(t), t \geq 0\}$ can be decomposed as follows

$$\psi(u) = \psi_d(u) + \psi_c(u),$$

where $\psi_d(u)$ is the probability of ruin that is caused by oscillation of the Brownian motion, meaning that that the surplus at the time of ruin is zero, and $\psi_c(u)$ is the probability that ruin is caused by a claim, and in this case the surplus at the time of ruin is negative.

Dufresne and Gerber (1991) defined the adjustment coefficient R as the strictly positive solution of the equation

$$-cr + \lambda\left[M_X(r) - 1\right] + \frac{1}{2}\sigma^2 r^2 = 0, \tag{2.3.33}$$

provided such a solution exists.

In this context, they derived an upper bound for the ruin probability as

$$\psi(u) \leq e^{-Ru} \text{ for } u \geq 0, \tag{2.3.34}$$

and an asymptotic formula is of the form $\psi(u) \sim Ce^{-Ru}$ as $u \to \infty$. An expression for C has also been given by Gerber (1970). A review of perturbed risk models and the Cramér-Lundberg approximations to ruin probabilities in these models can be found in Schlegel (1998).

2.4. Multivariate ruin models

Most practical models consist of more than one class of business. For example, in automobile insurance it is of interest to study the kind of dependence between annual claim numbers since it has an impact on the premium paid by the policyholder.

Consider an insurance company with $m \geq 1$ classes of business. The surplus process $\{U_i(t), t \geq 0\}$ of the i-th class of business is given by

$$U_i(t) = u_i + c_i\, t - \sum_{k=1}^{N_i(t)} X_{ik}, \quad t \geq 0, \quad i = 1, ..., m, \tag{2.4.1}$$

where

• the initial surplus and premium rate are denoted by u_i and c_i, respectively assuming that $u_i \geq 0$ and $c_i > 0$;

• the claim sizes are modeled by a sequence of independent and identically distributed positive random vectors, $\{(X_{1,k}, ..., X_{m,k})\}_{k \geq 1}$, independently of all $\{N_i(t), t \geq 0\}$, but allow $X_{1,k}, ..., X_{m,k}$ to be dependent for each $k \geq 1$;

• for $i = 1, ..., m$, $\{N_i(t), t \geq 0\}$ are possibly dependent counting processes with $N_i(t) < \infty$ almost surely for any fixed $t > 0$.

According to Definition 2.2.2, the time of ruin for the i-th class $(i = 1, ..., m)$ is defined as

$$\tau_i = \inf\{t \geq 0 : U_i(t) < 0\},$$

and the corresponding ruin probability as

$$\psi_i(u_i) = P(\tau_i < \infty \mid U_i(0) = u_i).$$

If for each $i = 1, ..., m$, the surplus $U_i(t) \geq 0$ for all $t \geq 0$ (no ruin occurs), we indicate this by writing $\tau_i = \infty$. To avoid the certainty of ruin, we assume that the net profit condition (2.2.4) is satisfied for each class of business, which in our case becomes

$$c_i - E[X_i] \lim_{t \to \infty} \frac{E[N_i(t)]}{t} > 0, \tag{2.4.2}$$

provided $E[X_i] < \infty$, for $i = 1, ..., m$.

Different ruin concepts for multivariate risk processes are introduced by Chan et al. (2003). For instance, the following types of times of ruin and the corresponding infinite-time ruin probabilities are defined as follows.

Definition 2.4.1. *1. The first time when ruin occurs in all classes simultaneously or at the same instant in time is defined by*

$$\tau_{sim} = \inf\{t \geq 0 : \max\{U_1(t), ..., U_m(t)\} < 0\}, \tag{2.4.3}$$

and the corresponding ruin probability by

$$\psi_{sim}(u_1, ..., u_m) = P\{\tau_{sim} < \infty | (U_1(0), ..., U_m(0)) = (u_1, ..., u_m)\}. \tag{2.4.4}$$

2. The first time when ruin occurs in all classes, but not necessarily simultaneously is defined by

$$\tau_{and} = \max(\tau_1, ..., \tau_m), \tag{2.4.5}$$

and the corresponding ruin probability by

$$\psi_{and}(u_1, ..., u_m) = P\{\tau_{and} < \infty | (U_1(0), ..., U_m(0)) = (u_1, ..., u_m)\}. \tag{2.4.6}$$

3. The first time when ruin occurs in at least one class of business is defined by

$$\tau_{or} = \min(\tau_1, ..., \tau_m), \tag{2.4.7}$$

and the corresponding ruin probability by

$$\psi_{or}(u_1, ..., u_m) = P\{\tau_{or} < \infty | (U_1(0), ..., U_m(0)) = (u_1, ..., u_m)\}. \tag{2.4.8}$$

4. The first time when the sum of $U_i(t)$, $i = 1, ..., m$, becomes negative is defined by

$$\tau_{sum} = \inf\{t \geq 0 : U_1(t) + ... + U_m(t) < 0\}, \tag{2.4.9}$$

and the corresponding ruin probability by

$$\psi_{sum}(u) = P\{\tau_{sum} < \infty | (U_1(0), ..., U_m(0)) = (u_1, ..., u_m)\}, \tag{2.4.10}$$

where $u = \sum_{j=1}^{m} u_j$. By convention, $\inf \emptyset = \infty$.

In the univariate case ($m = 1$) we have that

$$\psi_1(u_1) = \psi_{and}(u_1) = \psi_{or}(u_1) = \psi_{sim}(u_1) = \psi_{sum}(u_1),$$

where u_1 is the initial capital.

Multivariate models of type (2.4.1) impose dependence between the classes of business by introducing interaction between the number of claims and/or between the claim sizes across classes of business.

In regard to the dependence between the number of claims, assume that m types of claims are generated by n processes, $n > m$. In general m (of the n) processes are claim processes, one for each type of claim. The remaining $(n-m)$ processes are occurrence processes and result in claims of multiple types. For instance, a common event such as a natural disaster may cause various kinds of insurance claims (health, house, vehicle, etc.). One way to model this dependence is to consider, for example, that the claim arrivals in the m-dimensional risk model (2.4.1) follow the Poisson model with common shocks, which assumes that in addition to the individual shocks, a common shock affects the m classes of business and that another common shock has an impact on each couple of classes. Mathematically,

$$\begin{aligned} N_1(t) = &\quad N_{11}(t) + N_{12}(t) + ... + N_{1m}(t) + N_{1...m}(t), \\ . &\qquad\qquad . \\ . &\qquad\qquad . \\ . &\qquad\qquad . \\ N_m(t) &\quad N_{mm}(t) + N_{1m}(t) + ... + N_{m-1\ m}(t) + N_{1...m}(t), \end{aligned} \tag{2.4.11}$$

where $\{N_{ij}(t), t \geq 0\}$, $(1 \leq i \leq j \leq m)$ and $\{N_{1...m}(t), t \geq 0\}$ are all mutually independent Poisson processes with parameters λ_{ij} and $\lambda_{1...m}$, respectively.

The approach to modeling dependent classes of business by incorporating a common component into each of the associated claim-number processes has been studied by many authors, for example, Ambagaspitiya (1998, 1999, 2003), and Wang (1998).

The dependence between claim sizes across classes of business leads to multivariate distributions. Of more interest and practical value are methods which construct multivariate models from known marginal distributions. For example, suppose it were known that losses have Pareto (or exponential) distribution and they could be combined into a multivariate distribution that introduces a degree of association between these Pareto distributed random variables. Among the methods available, the copula has received a lot of attention in the actuarial literature; see for example, Frees and Valdez (1998), Wang (1998) and Cossette et al. (2008, 2010).

The notion of copula is presented in the next subsection. For the numerical results reported in Sections 4.8 and 5.5, we model the dependence in claim sizes through copula techniques.

2.4.1. Copulas

This section contains a review of the concept of copula; for details on this topic, we refer to the books authored by Joe (1997) or Nelsen (2006).

One typical way of realizing dependence between n random variables in a mathematical model is to combine them through a copula, which, informally, is a multivariate distribution function with uniform marginals. A formal definition follows.

Definition 2.4.2. *A function $C : [0,1]^n \to [0,1]$, $(n \geq 2)$, is called a n-dimensional copula (briefly n-copula) if it satisfies the following conditions:*

1. $C(u_1, ..., u_n)$ is increasing in each component u_i;

2. $C(u_1, ..., u_{k-1}, 0, u_{k+1}, ..., u_n) = 0$ for all $u_i \in [0,1]$, $i \neq k$, $k = 1, ..., , n$;

3. $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ for all $u_i \in [0,1]$, $i = 1, ..., n$;

4. C is n-increasing, that is, for all $(a_1, ..., a_n)$, $(b_1, ..., b_n) \in [0,1]^n$ with $a_i \leq b_i$,

$$\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+...+i_n} C(x_{1,i_1}, ..., x_{n,i_n}) \geq 0,$$

where $x_{k,1} = a_k$ and $x_{k,2} = b_k$ for all $k \in \{1, ..., n\}$.

Remark 2.4.1. *For any n-copula, where $n \geq 3$, each k-margin of C is a k-copula, where $2 \leq k \leq n$.*

Definition 2.4.3. *A copula $C : [0,1]^n \to [0,1]$ is called absolutely continuous if, when considered as a joint distribution, it has a density $c(u_1, ..., u_n)$ with respect to the Lebesgue measure on $[0,1]^n$ expressed as*

$$c(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1 ... \partial u_n}.$$

Most of the applications in the field of copulas are based on Sklar's theorem [Sklar (1959)], which explains the role copulas play in the relationship between multivariate distribution functions and their univariate marginals. In the following, we present a version of this theorem in terms of random variables.

Theorem 2.4.1. *Let X_1,...,X_n be random variables defined on a common probability space with distribution functions F_1,...,F_n, respectively, and joint distribution function H. Then, there exists an n-copula C such that*

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \text{ for all } x \in \overline{\mathbb{R}}^n. \tag{2.4.12}$$

If F_1,...,F_n are all continuous, then C is unique. Otherwise, C is uniquely determined on $Ran(F_1)\times ... \times Ran(F_n)$, where for each $i = 1, ..., n$, $Ran(F_i)$ denotes the range of F_i and is defined as $Ran(F_i) = \{y \in [0, 1] : y = f(x) \text{ for some } x \in \overline{\mathbb{R}}\}$. Conversely, for a given copula C and marginals F_1,...,F_n we have that (2.4.12) defines a joint distribution with marginals F_i.

We will now provide some examples of copulas that will be used in this thesis.

Example 2.4.1. *The independent copula is specified by*

$$C(u_1, ..., u_n) = u_1 \cdot ... \cdot u_n, \tag{2.4.13}$$

which corresponds to independent random variables.

Example 2.4.2. *The bivariate Farlie-Gumbel-Morgenstern (FGM) copula function is specified by*

$$C(u_1, u_2) = u_1 u_2 \left[1 + \theta(1 - u_1)(1 - u_2)\right], \quad \theta \in [-1, 1].$$

Example 2.4.3. *The multivariate FGM n-copula function is specified by*

$$C(u_1, ..., u_n) = \prod_{i=1}^{n} u_i \left[1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < ... < j_k \le n} \theta_{j_1 ... j_k} \prod_{l=1}^{k} (1 - u_{j_l})\right],$$

with

$$1 + \sum_{k=2}^{n} \sum_{1 \le j_1 < ... < j_k \le n} \theta_{j_1 ... j_k} \prod_{l=1}^{k} \epsilon_{j_l} \ge 0,$$

for any choice of ϵ_1,...,ϵ_n in $\{-1, 1\}$.

Example 2.4.4. *The Clayton copula function is specified by*

$$C(u_1, ..., u_n) = \left(u_1^{-\theta} + ... + u_n^{-\theta} - n + 1\right)^{-1/\theta}, \quad \theta > 0.$$

In the sequel, we give a review of the results concerning ruin probabilities in the context of multivariate risk models.

2.4.2. Review of the literature

As pointed out in Chan et al. (2003), ruin theory under multidimensional risk models is very complex. Even in a two-dimensional case, the problem is challenging. In ruin theory under multidimensional risk models, it is usually difficult to derive explicit results for the probability of ruin.

In our study, we focus on multivariate risk models with common shocks governed by Poisson or renewal processes and this motivates us to give a brief summary of the results related to these types of models.

The impact of dependence in the Poisson model with common shocks, characterized by (2.4.11), on the ruin probability of type ψ_{sum} has been discussed by many authors. For example, Cossette and Marceau (2000) examined the discrete-time risk models with three classes of correlated business modeled by, respectively, Poisson and negative binomial processes with common shocks. Under these settings, they focused on the risk process representing the total wealth of the company and showed that the probability of ruin ψ_{sum} increased and the adjustment coefficient R decreased under dependence when comparing the independent (without any common shocks) and dependent cases. This is further confirmed by Yuen and Wang (2002), Wang and Yuen (2005), who worked with a continuous-time model having a dependence structure specified by a thinning model applied to the claim number process. Here, if the stochastic sources related to claim occurrences of the m classes of business are classified into n groups, then it is assumed that each event in the k-th ($k = 1, ..., n$) group may cause a claim in the j-th class ($j = 1, ..., m$) with a certain probability.

We illustrate now the continuous-time process associated to the sum of surplus processes, where the dependence between claim arrivals is due to common shocks.

Consider the multivariate risk model (2.4.1) when $m = 2$ and the claim number processes $N_i(t)$ are correlated as in (2.4.11):

$$N_1(t) = N_{11}(t) + N_{12}(t) \text{ and } N_2(t) = N_{22}(t) + N_{12}(t),$$

but assuming that $N_{11}(t)$, $N_{22}(t)$, and $N_{12}(t)$ are independent renewal processes.

It is further assumed that for any $k \geq 1$, X_{1k} and X_{2k} are independent claim size random variables, and that they are independent of $N_1(t)$ and $N_2(t)$. For

$i = 1, 2$, let F_{X_i} and μ_{X_i} be the common distribution function and finite mean, respectively, of the claim sizes X_{ik}. Then the ruin probability ψ_{sum} defined by (2.4.10) is associated to the risk process:

$$U(t) = U_1(t) + U_2(t) = u + ct - \sum_{k=1}^{N_{11}(t)+N_{12}(t)} X_{1k} - \sum_{k=1}^{N_{22}(t)+N_{12}(t)} X_{2k}, \qquad (2.4.14)$$

where $u = u_1 + u_2 \geq 0$ and $c = c_1 + c_2 > 0$.

Yuen et al. (2002) examined the infinite-time ruin (survival) probability for the risk process (2.4.14) by considering the following two cases:

Case 1: Three Poisson processes

In this case, $N_{11}(t)$, $N_{22}(t)$, and $N_{12}(t)$ are assumed to be Poisson processes with rates λ_{11}, λ_{22}, and λ_{12}, respectively. Yuen et al. (2002) showed that the sum of the two dependent risk processes $\{U_1(t), t \geq 0\}$ and $\{U_2(t), t \geq 0\}$ can be converted back to a univariate compound Poisson model as follows.

Proposition 2.4.1. *The surplus process $U(t)$ defined by* (2.4.14) *is distributed the same way as the process*

$$U'(t) = u + ct - \sum_{k=1}^{M(t)} X'_k,$$

where $M(t)$ is a Poisson process with parameter $\lambda = \lambda_{11} + \lambda_{22} + \lambda_{12}$ independent of $\{X'_k\}_{k\geq 1}$ and $\{X'_k\}_{k\geq 1}$ is a sequence of independent and identically distributed random variables with the common distribution function described as

$$F_{X'}(x) = \frac{\lambda_{11}}{\lambda} F_{X_1}(x) + \frac{\lambda_{22}}{\lambda} F_{X_2}(x) + \frac{\lambda_{12}}{\lambda} F_{X_1} * F_{X_2}(x).$$

Note that $F_{X_1} * F_{X_2}$ stands for the convolution of F_{X_1} and F_{X_2} and is defined as

$$F_{X_1} * F_{X_2}(x) = \int_0^x F_{X_1}(x - y) dF_{X_2}(y). \qquad (2.4.15)$$

For the proof of this result we refer to Yuen et al. (2002).

Since the transformed process $U'(t)$ and the risk process $U(t)$ are identically distributed, the process $U(t)$ can be examined via $U'(t)$. Therefore, the risk process $U'(t)$ represents the classical model presented in Subsection 2.3.1 and ruin probability results related to this model can be applied.

Case 2: Dependent Poisson-Erlang case

In this case, for the risk model given by (2.4.14) it is assumed that the processes $\{N_{11}(t), t \geq 0\}$ and $\{N_{22}(t), t \geq 0\}$ are Poisson processes with rates λ_{11}, and λ_{22}, respectively, while the process $\{N_{12}(t), t \geq 0\}$ is an Erlang(2) process with parameter λ_{12}, meaning that the renewal claim inter-arrival times $\{T_i\}_{i\geq 1}$ follow an Erlang (2, λ_{12}) distribution, given by Definition 1.1.1, with common mean $E[T_i] = 2/\lambda_{12}$.

In the same manner as in the previous case, Yuen et al. (2002) showed that the risk process (2.4.14) can be converted to a risk process with two independent claim number processes as follows.

Proposition 2.4.2. *The surplus process* $U(t)$ *defined by* (2.4.16) *is distributed as the process*

$$U'(t) = u + ct - \sum_{k=1}^{M(t)} X'_k - \sum_{k=1}^{N_{12}(t)} Y'_k, \tag{2.4.16}$$

where $M(t) = N_{11}(t) + N_{22}(t)$ *is a Poisson process with parameter* $\lambda_{11} + \lambda_{22}$ *and* $\{X'_k\}_{k\geq 1}$, $\{Y'_k\}_{k\geq 1}$, $\{M(t), t \geq 0\}$, $\{N_{12}(t), t \geq 0\}$ *are all mutually independent. Furthermore,* $\{X'_k\}_{k\geq 1}$ *and* $\{Y'_k\}_{k\geq 1}$ *respectively, are independent and identically distributed random variables with distribution functions given by*

$$F_{X'}(x) = \frac{\lambda_{11}}{\lambda_{11} + \lambda_{22}} F_{X_1}(x) + \frac{\lambda_{22}}{\lambda_{11} + \lambda_{22}} F_{X_2}(x) \text{ and } F_{Y'}(x) = F_{X_1} * F_{X_2}(x),$$

where $F_{X_1} * F_{X_2}$ *is defined by* (2.4.15).

Thus, as in the previous case, the process $U(t)$ can be examined via $U'(t)$. In this sense, Yuen et al. (2002) derived an explicit expression for the survival probability $\phi(u) = 1 - \psi(u)$ associated to the risk process (2.4.16) assuming exponential claim sizes and the net profit condition $c > (\lambda_{11} + \lambda_{22})E[X'] + \lambda_{12}E[Y']/2$.

Proposition 2.4.3. *If the claim sizes* X_1 *and* X_2 *are exponentially distributed with equal mean* μ*, then the survival probability associated to the risk process* $\{U'(t), t \geq 0\}$*, defined by* (2.4.16)*, is given by*

$$\phi(u) = 1 + C_2 A(z_2) e^{z_2 u} + C_3 A(z_3) e^{z_3 u} + C_4 A(z_4) e^{z_4 u},$$

where

$$A(z) = 1 + \tfrac{\mu}{\lambda_{12}} \left(\lambda + \lambda_{12} - \tfrac{c}{\mu} \right) z + \tfrac{\mu^2}{\lambda_{12}} \left(\lambda - \tfrac{2c}{\mu} \right) z^2 - \tfrac{c\mu^2}{\lambda_{12}} z^3,$$

with $\lambda = \lambda_{11} + \lambda_{22} + \lambda_{12}$, $z_2 = -\mu^{-1}$, $z_3 = (c\mu)^{-1}(\lambda\mu - c)$,

$$z_4 = \tfrac{1}{2c\mu} \left[\lambda\mu - c - \left(8c\mu\lambda_{12} + (c - \lambda\mu)^2 \right)^{1/2} \right],$$

and the coefficients C_2, C_3, and C_4 can be computed by the following equations

$$\lambda = (cz_2 - \lambda)C_2 + (cz_3 - \lambda)C_3 + (cz_4 - \lambda)C_4,$$

$$\tfrac{\lambda_2}{\mu} = \left(cz_2^2 - \lambda - \tfrac{\lambda_{12}-c}{\mu}\right)C_2 + \left(cz_3^2 - \lambda - \tfrac{\lambda_{12}-c}{\mu}\right)C_3 + \left(cz_4^2 - \lambda - \tfrac{\lambda_{12}-c}{\mu}\right)C_4,$$

$$\lambda_{11} + \lambda_{22} =$$

$$(cz_2 - \lambda + \lambda_{12})\,A(z_2)C_2 + (cz_3 - \lambda + \lambda_{12})\,A(z_3)C_3 + (cz_4 - \lambda + \lambda_{12})\,A(z_4)C_4.$$

Thus, the problem involving the ruin probability of type ψ_{sum} can be reduced to a one-dimensional ruin problem and therefore, it is relatively easy to analyze.

In the literature there are very few exact formulas for ruin probabilities of type ψ_{or}, ψ_{and}, or ψ_{sim}. In this regard, for the bivariate risk model (2.4.1) where $N_1(t) \equiv N_2(t) \equiv N(t)$ and $N(t)$ is a Poisson process with rate λ, Chan et al. (2003) obtained an expression for the Laplace transform for the survival probability $\phi_{or}(u_1, u_2) = 1 - \psi_{or}(u_1, u_2)$. As they claimed, it is difficult to derive explicit expressions for $\phi_{or}(u_1, u_2)$ by inverting its Laplace transform, even in the case when the claims are exponentially distributed. In respect with this result, Dang et al. (2009) derived a recursive relationship for this type of survival probability in the case of exponential claim sizes using results from the theory of partial differential equations.

The latter result was recovered by Gong et al. (2012) providing recursive integral formulas for the survival probabilities of type ϕ_{or} in a multidimensional risk model defined as follows.

For $i = 1, ..., m$, the surplus process of the i-th class of business is given by

$$U_i(t) = u_i + c_i t - \sum_{k=1}^{N_{ii}(t)} Y_{ik} - \sum_{k=1}^{N_c(t)} Z_{ik}, \quad t \geq 0, \tag{2.4.17}$$

with the associated initial capital and premium rate denoted by $u_i = U_i(0) \geq 0$ and $c_i > 0$, respectively. The counting processes $\{N_{11}(t), t \geq 0\}$,...,$\{N_{mm}(t), t \geq 0\}$ and $\{N_c(t), t \geq 0\}$ are Poisson processes with rates λ_{11},...,λ_{mm} and λ_c, respectively. For a given $i = 1, ..., m$, $\{Y_{ik}\}_{k\geq 1}$ forms an independent and identically distributed (i.i.d.) sequence of positive random variables with common density function f_{ii}. Furthermore, $\{(Z_{1k}, ..., Z_{mk})\}_{k\geq 1}$ is a sequence of i.i.d. m-dimensional positive random vectors with common joint density function f_c. It is further assumed that $\{N_{ii}(t), t \geq 0\}$, for $i = 1, ..., m$, $\{N_c(t), t \geq 0\}$, $\{Y_{ik}\}_{k\geq 1}$, for $i = 1, ..., m$, and $\{(Z_{1k}, ..., Z_{mk})\}_{k\geq 1}$ are all mutually independent.

The n-th claim event arrival time is defined as $S = \sum_{k=1}^{n} V_k$, where $\{V_k\}_{k\geq 1}$ are the inter-arrival times corresponding to the Poisson process $\{\sum_{i=1}^{m} N_{ii}(t)+N_c(t), t \geq 0\}$ and hence i.i.d. exponential random variables each with mean $1/\lambda_s$, where

$$\lambda_s = \sum_{i=1}^{m} \lambda_{ii} + \lambda_c.$$

The following proposition formulated by Gong et al. (2012) gives a recursive integral relation satisfied by the probability that all $U_1(t)$,...,and $U_m(t)$ survive up to and including the n-th claim, denoted by $\phi_n(u_1, ..., u_m)$.

Proposition 2.4.4. *Assuming the risk model defined by* (2.4.20), *then*

$$\begin{aligned}\phi_{n+1}(u_1, ..., u_m) = \sum_{i=1}^{m} \int_0^\infty \int_0^{u_i+c_i t} & \phi_n(u_1 + c_1 t, ..., u_{i-1} + c_{i-1}t, u_i + c_i t - x_i, \\ & u_{i+1} + c_{i+1}t, ..., u_m + c_m t) f_{ii}(x_i)\lambda_{ii} e^{-\lambda_s t} dx_i dt \\ + \int_0^\infty \int_0^{u_m+c_m t} \cdots \int_0^{u_1+c_1 t} & \phi_n(u_1 + c_1 t - x_1, ..., u_m + c_m t - x_m) \\ & \times f_c(x_1, ..., x_m)\lambda_c e^{-\lambda_s t} dx_1 ... dx_m dt,\end{aligned}$$

with the starting point $\phi_0(u_1, ..., u_m) = 1$.

Therefore, the survival probability for τ_{or}, $\phi_{or}(u_1, ..., u_m) = 1-\psi_{or}(u_1, ..., u_m)$, is given by the limit

$$\phi_{or}(u_1, ..., u_m) = \lim_{n\to\infty} \phi_n(u_1, ..., u_m).$$

Motivated by the difficulty and instability in obtaining numerical results from the direct application of the recursive relation obtained in Proposition 2.4.4, Gong et al. (2012) studied two bivariate cases under the assumption of exponential or mixture of Erlang claims. They show that the recursive integrals can be reduced to recursive sums which are computationally more tractable. Furthermore, for the bivariate case, Gong et al. (2012) derived a result for the survival probability corresponding to τ_{and} as follows.

Proposition 2.4.5. *The survival probability* $\phi_{and}(u_1, u_2)$ *is*

$$\phi_{and}(u_1, u_2) = \lim_{n\to\infty} \left[\phi_n^1(u_1) + \phi_n^2(u_2) - \phi_n(u_1, u_2)\right],$$

where $\phi_n^1(u_1)$ *satisfies*

$$\begin{aligned}\phi_{n+1}^1(u_1) = \int_0^\infty \int_0^{u_1+c_1 t} & \phi_n^1(u_1 + c_1 t - x_1) \\ \times \left\{ f_{11}(x_1)\lambda_{11} + \left[\int_0^\infty f_c(x_1, x_2)dx_2\right] \lambda_c \right\} & e^{-\lambda_s t} dx_1 dt\end{aligned}$$

$$+\int_0^\infty \phi_n^1(u_1+c_1t)\lambda_{22}e^{-\lambda_s t}dt,$$

a similar relation for $\phi_{n+1}^2(u_2)$ *is obtained by reversing the roles of lines* 1 *and* 2 *and* $\phi_n(u_1,u_2)$ *is given by Proposition* 2.4.4.

In what follows, we present some of results on bounds, approximations or asymptotics for ruin probabilities in a multivariate setting.

Proposition 2.4.6. *Consider the bivariate risk model in* (2.4.1)*, where* $N_1(t)\equiv N_2(t)\equiv N(t)$ *and* $N(t)$ *is a Poisson process with rate* λ*. Then*

$$\psi_{or}(u_1,u_2)\geq \max\{\psi_1(u_1),\psi_2(u_2)\} \text{ and } \psi_{sim}(u_1,u_2)\leq \min\{\psi_1(u_1),\psi_2(u_2)\}.$$

For a proof, we refer to Chan et al. (2003).

The results of Proposition 2.4.6 are generalized and complemented by various stochastic bounds in Cai and Li (2005, 2007) for the multivariate risk model (2.4.1), where the number of claims for each class of business is modeled by a Poisson process $N(t)$ with rate λ, that is, $N_i(t)\equiv N(t)$ $(i=1,...,m)$, and the claims are stochastically dependent. In this sense, we first give the following definition.

Definition 2.4.4. *Let* $\boldsymbol{X}=(X_1,...,X_m)$ *be a real random vector.*

1. $\boldsymbol{X}$ *is said to be positively associated if*

$$E[f(\boldsymbol{X})g(\boldsymbol{X})]\geq E[f(\boldsymbol{X})]E[g(\boldsymbol{X})]$$

for any real increasing functions f*,* g *defined on* $\mathbb{R}^m$*.*

2. $\boldsymbol{X}$ *is said to be supermodular dependent if*

$$(X_1,...,X_m)\geq_{sm}(X_1^I,...,X_m^I),$$

where X_1^I,...,X_m^I *are independent, and* X_j^I *and* X_j*,* $1\leq j\leq m$*, have the same distribution.*

Proposition 2.4.7. *(1) For the multivariate compound Poisson risk model defined by* (2.4.1) *with a Poisson arrival process and positively associated claim vector, we have*

$$\prod_{j=1}^m \psi_j(u_j)\leq \psi_{and}(u_1,...,u_m)\leq \psi_{or}(u_1,...,u_m)\leq 1-\prod_{j=1}^m\left(1-\psi_j(u_j)\right). \quad (2.4.18)$$

(2) The same bounds from (2.4.18) *still hold when the claim size vector possesses the supermodular dependence.*

For the bivariate compound Poisson model with common shock obtained from (2.4.1), considering

$$N_1(t) = N_{11}(t) + N_{12}(t) \text{ and } N_2(t) = N_{22}(t) + N_{12}(t),$$

and assuming independent claim sizes across classes, Yuen et al. (2006) derived bounds for the infinite-time ruin probability $\psi_{or}(u_1, u_2)$, namely

$$\max\{\psi_1(u_1), \psi_2(u_2)\} \leq \psi_{or}(u_1, u_2) \leq \psi_1(u_1) + \psi_2(u_2) - \psi_1(u_1)\psi_2(u_2),$$

where the final expression is exactly the ruin probability in the case where $U_1(t)$ and $U_2(t)$ are independent. Also, an approximation method for computing the bivariate survival probability $\phi_{or}(u_1, u_2, t) = 1 - \psi_{or}(u_1, u_2, t)$ using a discrete bivariate compound binomial model was derived by Yuen et al. (2006).

Li, Liu and Tang (2007) investigated the ruin probability $\psi_{sim}(u_1, u_2)$ for the bivariate risk model perturbed by a diffusion process where the claim counting processes are assumed to be the same Poisson process with rate λ. This model is described by the following two-dimensional surplus process

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 + c_1 t - \sum_{k=1}^{N(t)} X_{1k} + \sigma_1 B_1(t) \\ u_2 + c_2 t - \sum_{k=1}^{N(t)} X_{2k} + \sigma_2 B_2(t) \end{pmatrix}, \quad t \geq 0, \tag{2.4.19}$$

where $(B_1(t), B_2(t))$ denotes a standard bidimensional Brownian motion with constant correlation coefficient $r \in [-1, 1]$, while $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ denote the volatility coefficients of $B_1(t)$ and $B_2(t)$, respectively.

Using martingale techniques, they obtained a Lundberg type upper bound for the infinite-time ruin probability $\psi_{sim}(u_1, u_2)$. More specifically, provided the set

$$G = \{(s_1, s_2) \mid s_1 \geq 0,\ s_2 \geq 0,\ M(s_1, s_2) < \infty\} \setminus \{(0, 0)\}$$

is non-empty, where $M(s_1, s_2) = E[e^{(s_1 X_1 + s_2 X_2)}]$ is the joint moment generating function of (X_1, X_2), they proved that the process

$$M(t) = e^{(-s_1 U_1(t) - s_2 U_2(t) - f(s_1, s_2) t)},\ t \geq 0,$$

is a martingale with respect to the natural filtration of $\{(U_1(t), U_2(t)),\ t \geq 0\}$, where

$$f(s_1, s_2) = \lambda M(s_1, s_2) - \lambda - c_1 s_1 - c_2 s_2 + \frac{1}{2}[\sigma_1^2 s_1^2 + 2r\sigma_1\sigma_2 s_1 s_2 + \sigma_2^2 s_2^2]. \tag{2.4.20}$$

The upper bound for the ruin probability $\psi_{sim}(u_1, u_2)$ is given by the following proposition.

Proposition 2.4.8. *Consider the bivariate risk model defined by relation* (2.4.19) *and* $f(s_1, s_2)$ *given by* (2.4.20). *If* $\sup_{(s_1,s_2)\in G} f(s_1, s_2) > 0$, *then*

$$\psi_{sim}(u_1, u_2) \leq \inf_{(s_1,s_2)\in\Delta} e^{-s_1u_1-s_2u_2},$$

where $\Delta = \{(s_1, s_2) \in G \mid f(s_1, s_2) = 0\}$.

The proof of this result can be found in Li, Liu and Tang (2007).

Furthermore, in the case of independent heavy-tailed claims an asymptotic estimate for the finite-time ruin probability was derived.

Proposition 2.4.9. *Assume that the components of both the claim vector* (X_1, X_2) *and the bidimensional Brownian motion* $(B_1(t), B_2(t))$ *are independent. If the claim size distributions are respectively,* F_1 *and* F_2 *belonging to the subexponential class, then, for each fixed time* $t > 0$,

$$\psi_{sim}(u_1, u_2; t) \sim \lambda t(\lambda t + 1)\overline{F}_1(u_1)\overline{F}(u_2),\ u_1 \to \infty,\ u_2 \to \infty.$$

This result is stated as Theorem 4.1 in Li, Liu and Tang (2007).

In a similar manner, Asmussen and Albrecher (2010) derived an upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$ assuming the special case of the multivariate risk model (2.4.1) for which $N_i(t) \equiv N(t)$, where $N(t)$ is a Poisson process with rate λ. This result is presented in the following proposition.

Proposition 2.4.10. *Consider the multivariate risk model given by* (2.4.1), *where* $N_i(t) \equiv N(t)$ $(i = 1, ..., m)$, *with* $N(t)$ *being a Poisson process with rate* λ. *If*

$$G = \{(s_1, ..., s_m) \mid s_1 > 0, .., \ s_m > 0,\ M(s_1, ..., s_m) = E[e^{s_1X_1+...+s_mX_m}] < \infty\},$$

and $\sup_{(s_1,...,s_m)\in G} f(s_1, ..., s_m) > 0$, *with* $f(s_1, ..., s_m)$ *defined by*

$$f(s_1, ..., s_m) = -\sum_{i=1}^{m} c_i s_i + \lambda\left[M(s_1, ..., s_m) - 1\right],$$

then

$$\psi_{sim}(u_1, ..., u_m) \leq \inf_{(s_1,...,s_m)\in\Delta} e^{-s_1u_1-...-s_mu_m},$$

where $\Delta = \{(s_1, ..., s_m) \in G \mid f(s_1, ..., s_m) = 0\}$.

Motivated and inspired by all of these results, we continue the investigation of ruin probabilities in a multivariate risk model setting. More specifically, since the results presented by Proposition 2.4.10 characterize a particular portfolio of classes of business, where claims occur only due to common shocks, we extend these results for the case where the claims are characterized by a Poisson model with common shocks, defined by (2.4.11), that illustrates more realistic situations. By adding a multivariate diffusion process, we further obtain upper bounds of Lundberg-type for the ruin probability of type ψ_{sim}. The results obtained for these models will be presented in Chapter 4. Moreover, in Chapter 5, we extend these multivariate models to the multivariate models where the correlation between the classes of business is due to a common renewal shock.

Throughout Chapters 4 and 5 we adopt the approach of deriving suitable martingales using tools from piecewise deterministic Markov processes theory.

Chapter 3

A CLASS OF BIVARIATE ERLANG (BVER) DISTRIBUTIONS

3.1. Introduction

Great interest has been shown in developing models that represent dependence for multivariate survival data or multivariate failure-time data that can be applied in various fields such as life insurance, demographic studies of the dynamics of mortality, economy, ecology as well as in the medical field.

The multivariate reduction technique is a popular and classical technique used for constructing dependent variables, both in the continuous and discrete cases. We present this technique in a fairly general form.

Let $\mathbf{T} = (T_0, T_1, ..., T_n)$ be an $(n+1)$-variate random vector with corresponding mutually independent cumulative distribution functions $F_i(y, \theta_i)$, i = 0,1, ..., n, and let $\mathbf{X} = (X_1, ..., X_n)$ be another random vector. Denote by L a functional mapping from R^{n+1} to R^n, such that $\mathbf{X} = L(\mathbf{T})$.

Definition 3.1.1. *The random vector $\boldsymbol{X}$ is said to posses the cumulative distribution function $F(x_1, ..., x_n, \theta^*)$ parametrized by the vector $\theta^* = (\theta_1^*, \theta_2^*, ..., \theta_n^*)$, such that*

$$\theta_j^* = \eta_j(\theta_0, \theta_1, ..., \theta_n)$$

for specific functions η_j, $j = 1, 2, ..., n$.

Some examples of mapping $\mathbf{X} = L(\mathbf{T})$ are $(X_1, ..., X_n)$=$(\min(T_0, T_1), ..., \min(T_0, T_n))$, or $(X_1, ..., X_n) = (\max(T_0, T_1), ..., \max(T_0, T_n))$.

Our attention focuses on the bivariate case, when the method described by Definition 3.1.1 is known as the trivariate reduction technique. A relevant model obtained by using a special case of the trivariate reduction technique is the bivariate survival model of Marshall-Olkin type. This model is described in its general form as follows.

Let T_0, T_1 and T_2 be mutually independent continuous positive random variables with the corresponding survival functions $\overline{F}_{T_i}(t) = P(T_i > t)$, $i = 0, 1, 2$. If X and Y are defined as

$$X = \min(T_1, T_0) \quad \text{and} \quad Y = \min(T_2, T_0),$$

then the bivariate distribution of the random vector (X, Y) is defined for $x, y > 0$ by the following joint survival function

$$\begin{aligned} \overline{F}_{X,Y}(x, y) &= P(X > x, Y > y) = P(T_1 > x, T_2 > y, T_0 > \max(x, y)) \\ &= \overline{F}_{T_1}(x)\overline{F}_{T_2}(y)\overline{F}_{T_0}(\max(x, y)), \end{aligned} \tag{3.1.1}$$

in view of the mutual independence of T_0, T_1 and T_2.

Therefore, the random variables X and Y are dependent through the common random latent variable T_0. Mardia (1970) showed that this technique leads to positive correlation between X and Y, that is, the correlation coefficient

$$\rho(X, Y) = \frac{E[(X - E(X))(Y - E(Y))]}{\sqrt{Var[X]}\sqrt{Var[Y]}} = \frac{Cov(X, Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}} > 0,$$

which means that Y tends to increase as X increases.

In this dissertation, our contribution is to introduce a class of bivariate Erlang distributions by assuming that the random variables T_1, T_2 and T_0 follow different Erlang distributions. As mentioned in Chapter 1, the Erlang distribution is a special case of the gamma distribution with positive integer shape parameter, while the mixture of Erlang distributions can be used to approximate any positive continuous distribution [Tijms (1994)], a property illustrated by relation (1.2.2). We obtain that the bivariate exponential (BVE) distribution with exponential marginals introduced by Marshall and Olkin (1967a) and presented in Section 1.4 is a particular case of the bivariate Erlang distribution proposed in this thesis.

We establish that the marginals X, Y and $\min(X, Y)$ follow finite mixtures of Erlang distributions, while $\max(X, Y)$ follows a weighted average of Erlang distributions. Then, we consider the case where the components T_i, $i = 0, 1, 2$ have finite mixture of Erlang distributions.

We show that these bivariate distributions have potential applications in life-insurance and finance.

This chapter is structured as follows.

In Section 3.2, a review of other bivariate distributions of Marshall-Olkin type defined by (3.1.1) is presented.

In Section 3.3, we introduce the new bivariate distribution of Marshall-Olkin type based on Erlang distributions. This distribution is a mixture of an absolutely continuous distribution and a singular part concentrating its mass on the diagonal line $x = y$, a property that is illustrated in Subsection 3.3.1. We complete Section 3.3 by deriving the joint probability density function along with the marginal distributions, conditional probability density functions and conditional expectations.

The Laplace transform of the distribution, the moments and correlation structure are given in Section 3.4.

In Section 3.5, this distribution is extended to a bivariate distribution of Marshall-Olkin type based on finite mixtures of Erlang distributions and this way, a mixture of bivariate Erlang distributions is obtained.

The usefulness of these bivariate distributions in finance and insurance is presented in Section 3.6.

We adopt an Expectation-Maximization (EM) algorithm, which was proposed by Karlis (2003), to compute the maximum likelihood (ML) estimators for the bivariate Erlang distribution for the case where the shape parameters are known. The method and simulation results are illustrated in Section 3.7. We draw conclusions from our work in Section 3.8.

3.2. Review of the literature

Along the same line as the BVE model, Marshall and Olkin (1967a) also suggested the bivariate Weibull distribution, where the marginals are Weibull distributed. For this, if the random variables $T_i \sim \text{Weibull}(\alpha, \lambda_i)$, $i = 0, 1, 2$, with survival functions

$$\overline{F}_{T_i}(t) = e^{-\lambda_i t^{\alpha}}, \ t \geq 0, \ \lambda_i > 0, \ \alpha > 0,$$

then the bivariate vector (X, Y) has a bivariate Weibull distribution with parameters α, λ_0, λ_1, and λ_2 and the joint survival function defined by (3.1.1), while X follows a Weibull$(\alpha, \lambda_0 + \lambda_1)$ distribution and Y follows a Weibull$(\alpha, \lambda_0 + \lambda_2)$ distribution. The bivariate Weibull model is more flexible than the BVE model because of the presence of the shape parameter α.

Karlis (2003) developed an Expectation-Maximization (EM) algorithm for the computation of the maximum likelihood estimators of the BVE distribution, while Kundu and Dey (2009) using a different EM algorithm from that of Karlis, derived the maximum likelihood estimators for the bivariate Weibull distribution.

Based on Marshall-Olkin's survival model defined by (3.1.1), we list the following bivariate distributions:

• The bivariate Pareto distribution [Veenus and Nair (1994)], where the random variables T_i, $i = 0, 1, 2$, follow Pareto distributions and have survival functions defined as

$$\overline{F}_{T_i}(t) = \left(\frac{t}{\beta}\right)^{-\lambda_i}, \quad t \geq \beta, \quad \lambda_i > 0.$$

• The bivariate Gompertz distribution [Al-Khedhairi and El-Gohary (2008)], where the random variables T_i follow Gompertz distributions and the random variable T_0 follows an exponential distribution with survival functions given by

$$\overline{F}_{T_i}(t) = e^{-\frac{\alpha_i}{\beta_i}(e^{\beta_i t} - 1)}, \quad t \geq 0, \ \alpha_i > 0, \ \beta_i > 0, \quad i = 1, 2,$$

$$\text{and} \quad \overline{F}_{T_0}(t) = e^{-\theta t}, \ t \geq 0, \ \theta > 0.$$

• The bivariate distribution based on the generalized exponential and exponential distributions [Sarhan and Balakrishnan (2007)], where the random variables T_i follow generalized exponential distributions and the random variable T_0 follows an exponential distribution with survival functions given by

$$\overline{F}_{T_i}(t) = 1 - (1 - e^{-t})^{\theta_i}, \quad t \geq 0, \ \theta_i > 0, \quad i = 1, 2,$$

$$\text{and} \quad \overline{F}_{T_0}(t) = e^{-\theta_0 t}, \ t \geq 0, \ \theta_0 > 0.$$

• Later on, Kundu and Gupta (2010) modified the bivariate distribution of Sarhan and Balakrishnan (2007) by considering different generalized exponential distributions for the components T_i, $i = 0, 1, 2$, that is,

$$\overline{F}_{T_i}(t) = 1 - (1 - e^{-\lambda t})^{\theta_i}, \quad t \geq 0, \ \theta_i > 0, \ \lambda > 0,$$

and also provided the maximum likelihood estimators of the unknown parameters using an EM algorithm.

In addition to these distributions, we propose a new class of bivariate distributions using Erlang distributions in the survival model given by (3.1.1).

3.3. The bivariate Erlang distribution

In this section, we define the bivariate Erlang distribution of Marshall-Olkin type and compute its joint survival function. This is followed by a representation of the joint survival function as a mixture of an absolute continuous part and a singular part. The joint probability density function (p.d.f.), marginal distributions, conditional probability density function and expectation are formulated in this section. Also, the distributions for the minimum and maximum of the marginals are derived.

3.3.1. The joint survival function

Let us assume that T_0, T_1 and T_2 are mutually independent random variables having Erlang distributions with parameters (k_0, λ_0), (k_1, λ_1) and (k_2, λ_2), respectively, with $\lambda_i > 0$, for $i = 0, 1, 2$, and k_i is a positive integer for each i. Given these assumptions, the probability density function of T_i, for $i = 0, 1, 2$, is of the form

$$f_{T_i}(t) = \frac{\lambda_i^{k_i} t^{k_i - 1} e^{-\lambda_i t}}{(k_i - 1)!}, \quad t \geq 0, \tag{3.3.1}$$

and the survival function is given by

$$\overline{F}_{T_i}(t) = e^{-\lambda_i t} \sum_{n=0}^{k_i - 1} \frac{(\lambda_i t)^n}{n!}, \quad t \geq 0. \tag{3.3.2}$$

In this context, the random vector (X, Y), where

$$X = \min(T_1, T_0) \ \text{ and } \ Y = \min(T_2, T_0),$$

is said to follow the *bivariate Erlang (BVEr) distribution.*

The joint survival function $\overline{F}_{X,Y}(x,y)$ of the random vector (X,Y) is illustrated by the following proposition.

Proposition 3.3.1. *For $x,y \geq 0$, the bivariate Erlang distribution is given by*

$$\overline{F}_{X,Y}(x,y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x,y)} \sum_{n=0}^{k_1-1} \frac{(\lambda_1 x)^n}{n!} \sum_{n=0}^{k_2-1} \frac{(\lambda_2 y)^n}{n!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 \max(x,y))^n}{n!}.$$

Proof. The result is obtained by substituting survival functions of T_i, for $i = 0,1,2$, given by (3.3.2) into relation (3.1.1). □

Note that when $k_0 = k_1 = k_2 = 1$, for each $i = 0,1,2$, T_i is exponentially distributed with parameter λ_i, and in this case the joint survival function of (X,Y) given by Proposition 3.3.1 becomes

$$\overline{F}_{X,Y}(x,y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x,y)}, \; x,y > 0,$$

which describes the bivariate exponential (BVE) distribution with exponential marginals introduced by Marshall and Olkin (1967a) and presented in Section 1.4. Thus, the BVE distribution is a particular case of the bivariate Erlang distribution.

Proposition 3.3.2. *If (X,Y) follows a BVEr distribution with parameters (k_0, λ_0, k_1, λ_1, k_2, λ_2), then (aX, aY) follows a BVEr distribution with parameters (k_0, λ_0/a, k_1, λ_1/a, k_2, λ_2/a) for $a > 0$.*

Proof. The result is easily obtained by using the result of Proposition 3.3.1 in the following relation

$$\overline{F}_{aX,aY}(x,y) = P(aX > x, aY > y) = \overline{F}_{X,Y}(\frac{x}{a}, \frac{y}{a}) \text{ for } a > 0.$$

□

We introduce the following notations which will be used throughout this chapter. Let a_n, b_n, and c_n be the coefficients of the following polynomials

$$\sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} = \sum_{n=0}^{k_1+k_0-2} a_n x^n, \tag{3.3.3}$$

$$\sum_{n=0}^{k_2-1} \frac{\lambda_2^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} = \sum_{n=0}^{k_2+k_0-2} b_n x^n, \tag{3.3.4}$$

$$\sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n x^n}{n!} = \sum_{n=0}^{k_1+k_2-2} c_n x^n. \tag{3.3.5}$$

The BVEr distribution is not absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}_+^2$. It consists of an absolutely continuous part and a singular part on the diagonal $x = y \geq 0$, both of them being formulated in the following proposition.

Proposition 3.3.3. *The joint survival function of the BVEr distribution can be written as*

$$\overline{F}_{X,Y}(x,y) = \left(1 - \frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!}\right)\overline{F}_{ac}(x,y)$$
$$+\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!}\overline{F}_s(x,y),$$

where the singular distribution $\overline{F}_s(x,y)$ *is*

$$\overline{F}_s(x,y) = h(\max(x,y)),$$

the absolutely continuous distribution $\overline{F}_{ac}(x,y)$ *is given by*

$$\overline{F}_{ac}(x,y) = \frac{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!\overline{F}_{X,Y}(x,y) - \lambda_0^{k_0} h(\max(x,y))}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)! - \lambda_0^{k_0}},$$

with $h(x)$ *expressed as*

$$h(x) = e^{-(\lambda_1+\lambda_2+\lambda_0)x} \sum_{n=0}^{k_1+k_2-2} c_n \frac{(n+k_0-1)!}{(\lambda_1+\lambda_2+\lambda_0)^n} \sum_{l=0}^{n+k_0-1} \frac{(\lambda_1+\lambda_2+\lambda_0)^l x^l}{l!}.$$

Proof. Using the following relation

$$\overline{F}_{X,Y}(x,y) = P[X>x, Y>y, \{X=Y\}] + P[X>x, Y>y, \{X=Y\}^c],$$

and the representation of $\overline{F}_{X,Y}(x,y)$ as a mixture of both absolutely continuous and singular components, i.e.,

$$\overline{F}_{X,Y}(x,y) = (1-\alpha)\overline{F}_s(x,y) + \alpha\overline{F}_{ac}(x,y),\ 0 \leq \alpha \leq 1,$$

we obtain

$$(1-\alpha)\overline{F}_s(x,y) = P[X>x, Y>y, \{X=Y\}]$$

and

$$\alpha\overline{F}_{ac}(x,y)=\overline{F}_{X,Y}(x,y)-P[X>x,Y>y,\{X=Y\}].$$

To complete the proof we need to compute $P[X>x,Y>y,\{X=Y\}]$ and α. Therefore,

$$P[X>x,Y>y,\{X=Y\}]=P[X=Y>\max(x,y)]$$

$$=P[\min(T_1,T_0)=\min(T_2,T_0)>\max(x,y)]$$

$$=P[T_1\geq T_0,T_2\geq T_0,T_0>\max(x,y)]=\int\limits_{\max(x,y)}^{\infty}\overline{F}_{T_1}(t)\overline{F}_{T_2}(t)f_{T_0}(t)dt$$

$$=\int\limits_{\max(x,y)}^{\infty}\frac{e^{-(\lambda_1+\lambda_2+\lambda_0)t}\lambda_0^{k_0}t^{k_0-1}}{(k_0-1)!}\sum_{n=0}^{k_1-1}\frac{(\lambda_1 t)^n}{n!}\sum_{n=0}^{k_2-1}\frac{(\lambda_2 t)^n}{n!}dt$$

$$=\int\limits_{\max(x,y)}^{\infty}\frac{e^{-(\lambda_1+\lambda_2+\lambda_0)t}\lambda_0^{k_0}t^{k_0-1}}{(k_0-1)!}\sum_{n=0}^{k_1+k_2-2}c_n t^n dt$$

$$=\frac{\lambda_0^{k_0}}{(k_0-1)!}\sum_{n=0}^{k_1+k_2-2}c_n\int\limits_{\max(x,y)}^{\infty}e^{-(\lambda_1+\lambda_2+\lambda_0)t}t^{n+k_0-1}dt, \qquad (3.3.6)$$

where for $n=0,1,...,k_1+k_2-2$, c_n are given by (3.3.5). By using the result

$$\int\limits_{x}^{\infty}y^n e^{-\alpha y}dy=\frac{n!e^{-\alpha x}}{\alpha^{n+1}}\sum_{l=0}^{n}\frac{\alpha^l x^l}{l!}, \qquad (3.3.7)$$

relation (3.3.6) is equivalent to

$$=\frac{\lambda_0^{k_0}}{(k_0-1)!}\left[\sum_{n=0}^{k_1+k_2-2}\frac{e^{-(\lambda_1+\lambda_2+\lambda_0)\max(x,y)}c_n(n+k_0-1)!}{(\lambda_1+\lambda_2+\lambda_0)^{n+k_0}}\right]\times$$

$$\left[\sum_{l=0}^{n+k_0-1}\frac{(\lambda_1+\lambda_2+\lambda_0)^l(\max(x,y))^l}{l!}\right]$$

$$=\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!}h(\max(x,y)),$$

where $h(x)=e^{-(\lambda_1+\lambda_2+\lambda_0)x}\sum\limits_{n=0}^{k_1+k_2-2}c_n\frac{(n+k_0-1)!}{(\lambda_1+\lambda_2+\lambda_0)^n}\sum\limits_{l=0}^{n+k_0-1}\frac{(\lambda_1+\lambda_2+\lambda_0)^l x^l}{l!}$. Therefore, by taking $\alpha=1-\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!}$, the proof is completed. □

We note that when $k_0 = k_1 = k_2 = 1$, the result of Proposition 3.3.3 becomes the result obtained by Marshall and Olkin (1967a) for the BVE distribution, which is given by Proposition 1.4.1 from Section 1.4.

3.3.2. The marginals, minimum and maximum

We start by computing the marginal survival functions of the BVEr distribution. Therefore, for $x > 0$, we have

$$\overline{F}_X(x) = P(X > x) = P(T_1 > x)P(T_0 > x) = e^{-(\lambda_1+\lambda_0)x} \sum_{n=0}^{k_1-1} \frac{(\lambda_1 x)^n}{n!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 x)^n}{n!},$$

and

$$\overline{F}_Y(y) = P(Y > y) = P(T_2 > y)P(T_0 > y) = e^{-(\lambda_2+\lambda_0)y} \sum_{n=0}^{k_2-1} \frac{(\lambda_2 y)^n}{n!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 y)^n}{n!}.$$

Proposition 3.3.4. *The marginal density functions of X and Y are given by*

$$f_X(x) = e^{-(\lambda_1+\lambda_0)x} \left[\frac{\lambda_1^{k_1} x^{k_1-1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} + \frac{\lambda_0^{k_0} x^{k_0-1}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \right], \quad x > 0,$$

and

$$f_Y(y) = e^{-(\lambda_2+\lambda_0)y} \left[\frac{\lambda_2^{k_2} y^{k_2-1}}{(k_2-1)!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n y^n}{n!} + \frac{\lambda_0^{k_0} y^{k_0-1}}{(k_0-1)!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n y^n}{n!} \right], \quad y > 0.$$

Moreover, these marginals can be represented as finite mixtures of Erlang distributions with common rate parameters $\lambda_1 + \lambda_0$ and $\lambda_2 + \lambda_0$, respectively.

Proof. The results are easily obtained from

$$f_X(x) = -\frac{d}{dx}\overline{F}_X(x) = -\frac{d}{dx}\left(\overline{F}_{T_1}(x)\overline{F}_{T_0}(x)\right) = f_{T_1}(x)\overline{F}_{T_0}(x) + f_{T_0}(x)\overline{F}_{T_1}(x), \text{ and}$$

$$f_Y(y) = -\frac{d}{dy}\overline{F}_Y(y) = -\frac{d}{dy}\left(\overline{F}_{T_2}(y)\overline{F}_{T_0}(y)\right) = f_{T_2}(y)\overline{F}_{T_0}(y) + f_{T_0}(y)\overline{F}_{T_2}(y),$$

where $f_{T_i}(t)$ and $\overline{F}_{T_i}(t)$ $(i = 0, 1, 2)$ are given by (3.3.1) and (3.3.2), respectively. The density $f_X(x)$ can be rewritten as

$$f_X(x) = \sum_{n=0}^{k_0-1} \frac{\lambda_1^{k_1}\lambda_0^n (n+k_1-1)!}{(\lambda_1+\lambda_0)^{n+k_1}(k_1-1)!n!} \frac{e^{-(\lambda_1+\lambda_0)x}(\lambda_1+\lambda_0)^{n+k_1}x^{n+k_1-1}}{(n+k_1-1)!}$$

$$+ \sum_{n=0}^{k_1-1} \frac{\lambda_0^{k_0}\lambda_1^n (n+k_0-1)!}{(\lambda_1+\lambda_0)^{n+k_0}(k_0-1)!n!} \frac{e^{-(\lambda_1+\lambda_0)x}(\lambda_1+\lambda_0)^{n+k_0}x^{n+k_0-1}}{(n+k_0-1)!}$$

$$= \sum_{n=0}^{k_0-1} \alpha_n^{(1)} f_n^{(1)}(x) + \sum_{n=0}^{k_1-1} \alpha_n^{(2)} f_n^{(2)}(x), \tag{3.3.8}$$

where $\alpha_n^{(1)} = \frac{\lambda_1^{k_1}\lambda_0^n(n+k_1-1)!}{(\lambda_1+\lambda_0)^{n+k_1}(k_1-1)!n!}$, $\alpha_n^{(2)} = \frac{\lambda_0^{k_0}\lambda_1^n(n+k_0-1)!}{(\lambda_1+\lambda_0)^{n+k_0}(k_0-1)!n!}$, while $f_n^{(1)}(x)$ and $f_n^{(2)}(x)$ are the probability density functions for Erlang$(n+k_1, \lambda_1+\lambda_0)$ and Erlang$(n+k_0, \lambda_1+\lambda_0)$ distributions, respectively. Since $\int_0^\infty f_X(x)dx = 1$ and the same property holds for the densities $f_n^{(1)}(x)$ $(n = 0, 1, ..., k_0-1)$ and $f_n^{(2)}(x)$ $(n = 0, 1, ..., k_1-1)$, relation (3.3.8) yields

$$\sum_{n=0}^{k_0-1} \alpha_n^{(1)} + \sum_{n=0}^{k_1-1} \alpha_n^{(2)} = 1. \tag{3.3.9}$$

Hence, (3.3.8) represents a finite mixture of Erlang distributions with common rate parameter $\lambda_1+\lambda_0$, and the values of shape parameters range from $\min\{k_0, k_1\}$ to k_0+k_1-1, that is, $f_X(x)$ can be represented as

$$f_X(x) = \sum_{j=\min\{k_0,k_1\}}^{k_0+k_1-1} \alpha_j \text{Erlang}(j, \lambda_1+\lambda_0), \tag{3.3.10}$$

where α_j can be one of $\alpha_{n_1}^{(1)}$ $(n_1 = 0, 1, ..., k_0-1)$ and $\alpha_{n_2}^{(2)}$ $(n_2 = 0, 1, ..., k_1-1)$ if $n_1 \neq n_2$, and for the case $n_1 = n_2 = n$, α_j is equal to $\alpha_n^{(1)} + \alpha_n^{(2)}$. Therefore, $\alpha_j \geq 0$ for each j and, in view of (3.3.9),

$$\sum_{j=\min\{k_0,k_1\}}^{k_0+k_1-1} \alpha_j = 1. \tag{3.3.11}$$

By replacing k_1 by k_2 and λ_1 by λ_2 into expression (3.3.10), we obtain a similar result for the density of Y, that is, it can be represented as a finite mixture of Erlang distributions with common rate parameter $\lambda_2+\lambda_0$, and the values of shape parameters range from $\min\{k_0, k_2\}$ to k_0+k_2-1. We conclude that, similarly to $f_X(x)$, the probability density function of Y has the form

$$f_Y(y) = \sum_{j=\min\{k_0,k_2\}}^{k_0+k_2-1} \beta_j \text{Erlang}(j, \lambda_2+\lambda_0), \tag{3.3.12}$$

with $\beta_j \geq 0$ for each j and

$$\sum_{j=\min\{k_0,k_2\}}^{k_0+k_2-1} \beta_j = 1. \tag{3.3.13}$$

$\square$

Note that for the particular case, where $k_0 = k_1 = k_2 = 1$, the probability density functions obtained in Proposition 3.3.4 become

$$f_X(x) = (\lambda_1 + \lambda_0)e^{-(\lambda_1+\lambda_0)x} \quad \text{and} \quad f_Y(y) = (\lambda_2 + \lambda_0)e^{-(\lambda_2+\lambda_0)y},$$

which illustrate property (1) of Proposition 1.4.2 regarding the BVE distribution.

We remark that X and Y have identical marginals if and only if $\lambda_1 = \lambda_2$.

Proposition 3.3.5. *If (X, Y) follows a BVEr distribution with parameters (k_0, λ_0, k_1, λ_1, k_2, λ_2), then $\min(X, Y)$ follows a finite mixture of Erlang distributions with common rate parameter $\lambda_0 + \lambda_1 + \lambda_2$.*

PROOF. Let $Z = \min(X, Y)$. Since T_0, T_1, and T_2 are mutually independent, for $x > 0$ we have

$$\overline{F}_Z(x) = P(\min(X, Y) > x) = \overline{F}_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x). \tag{3.3.14}$$

By differentiating (3.3.14), the probability density function of Z is computed as

$$\begin{aligned} f_Z(x) &= f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x) + f_{T_1}(x)\overline{F}_{T_0}(x)\overline{F}_{T_2}(x) + f_{T_2}(x)\overline{F}_{T_0}(x)\overline{F}_{T_1}(x) \\ &= e^{-(\lambda_0+\lambda_1+\lambda_2)x}\frac{\lambda_0^{k_0}x^{k_0-1}}{(k_0-1)!}\sum_{n=0}^{k_1+k_2-2} c_n x^n + e^{-(\lambda_0+\lambda_1+\lambda_2)x}\frac{\lambda_1^{k_1}x^{k_1-1}}{(k_1-1)!}\sum_{n=0}^{k_0+k_2-2} b_n x^n \\ &\qquad + e^{-(\lambda_0+\lambda_1+\lambda_2)x}\frac{\lambda_2^{k_2}x^{k_2-1}}{(k_2-1)!}\sum_{n=0}^{k_0+k_1-2} a_n x^n, \end{aligned} \tag{3.3.15}$$

where a_n, b_n, and c_n are given by (3.3.3), (3.3.4) and (3.3.5), respectively. By similar arguments as in the proof of Proposition 3.3.4, (3.3.15) is equivalent to

$$\begin{aligned} &= \sum_{n=0}^{k_1+k_2-2} \frac{c_n\lambda_0^{k_0}(n+k_0-1)!}{(\lambda_0+\lambda_1+\lambda_2)^{n+k_0}} \cdot \frac{e^{-(\lambda_0+\lambda_1+\lambda_2)x}(\lambda_0+\lambda_1+\lambda_2)^{n+k_0}x^{n+k_0-1}}{(n+k_0-1)!} \\ &+ \sum_{n=0}^{k_0+k_2-2} \frac{b_n\lambda_1^{k_1}(n+k_1-1)!}{(\lambda_0+\lambda_1+\lambda_2)^{n+k_1}} \cdot \frac{e^{-(\lambda_0+\lambda_1+\lambda_2)x}(\lambda_0+\lambda_1+\lambda_2)^{n+k_1}x^{n+k_1-1}}{(n+k_1-1)!} \\ &+ \sum_{n=0}^{k_0+k_1-2} \frac{a_n\lambda_2^{k_2}(n+k_2-1)!}{(\lambda_0+\lambda_1+\lambda_2)^{n+k_2}} \cdot \frac{e^{-(\lambda_0+\lambda_1+\lambda_2)x}(\lambda_0+\lambda_1+\lambda_2)^{n+k_2}x^{n+k_2-1}}{(n+k_2-1)!} \\ &= \sum_{j=\min\{k_0,k_1,k_2\}}^{k_0+k_1+k_2-2} \gamma_j \text{Erlang}(j, \lambda_0+\lambda_1+\lambda_2), \end{aligned} \tag{3.3.16}$$

which represents a finite mixture of Erlang distributions with common rate parameter $\lambda_0+\lambda_1+\lambda_2$ and weights $\gamma_j \geq 0$ with

$$\sum_{j=\min\{k_0,k_1,k_2\}}^{k_0+k_1+k_2-2} \gamma_j = 1. \tag{3.3.17}$$

□

Proposition 3.3.6. *If* (X, Y) *follows a BVEr distribution with parameters* (k_0, λ_0, k_1, λ_1, k_2, λ_2), *then* $\max(X, Y)$ *is a weighted average of Erlang distributions.*

Proof. Let $W = \max(X, Y)$. Then, for $x > 0$, the distribution of W is

$$F_W(x) = P(\max(X, Y) \leq x) = P(X \leq x, Y \leq y) = 1 - \overline{F}_X(x) - \overline{F}_Y(x) + \overline{F}_Z(x), \tag{3.3.18}$$

where $Z = \min(X, Y)$. By differentiating (3.3.18), the probability density function of W is expressed as

$$f_W(x) = f_X(x) + f_Y(x) - f_Z(x). \tag{3.3.19}$$

Now, using (3.3.10), (3.3.12), and (3.3.16), from (3.3.19) it results that

$$f_W(x) = \sum_{j=\min\{k_0,k_1\}}^{k_0+k_1-1} \alpha_j \mathrm{Erlang}(j, \lambda_1+\lambda_0) + \sum_{j=\min\{k_0,k_2\}}^{k_0+k_2-1} \beta_j \mathrm{Erlang}(j, \lambda_2+\lambda_0)$$
$$- \sum_{j=\min\{k_0,k_1,k_2\}}^{k_0+k_1+k_2-2} \gamma_j \mathrm{Erlang}(j, \lambda_0+\lambda_1+\lambda_2),$$

where $\sum_{j=\min\{k_0,k_1\}}^{k_0+k_1-1} \alpha_j + \sum_{j=\min\{k_0,k_2\}}^{k_0+k_2-1} \beta_j - \sum_{j=\min\{k_0,k_1,k_2\}}^{k_0+k_1+k_2-2} \gamma_j = 1$, in view of (3.3.11), (3.3.13), and (3.3.17). Hence, $\max(X, Y)$ is a weighted average of Erlang distributions. □

3.3.3. The joint probability density function

The joint probability density function of X and Y is described in Proposition 3.3.7 below.

Proposition 3.3.7. *The joint probability density function of the BVEr distribution has the following form*

$$f_{X,Y}(x,y) = \begin{cases} \left[f_{T_1}(x)\overline{F}_{T_0}(x) + f_{T_0}(x)\overline{F}_{T_1}(x)\right] f_{T_2}(y), & \text{for } x > y > 0, \\ \left[f_{T_2}(y)\overline{F}_{T_0}(y) + f_{T_0}(y)\overline{F}_{T_2}(y)\right] f_{T_1}(x), & \text{for } y > x > 0, \\ f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x), & \text{for } x = y > 0. \end{cases}$$

Proof. In order to compute the joint probability density function of X and Y, we consider the following three cases: $x > y > 0$, $y > x > 0$, and $x = y > 0$. If $x > y > 0$, by relation (3.1.1) we have

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}\overline{F}_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}\left(\overline{F}_{T_1}(x)\overline{F}_{T_0}(x)\overline{F}_{T_2}(y)\right)$$

$$= \frac{\partial}{\partial y}\left[-f_{T_1}(x)\overline{F}_{T_0}(x)\overline{F}_{T_2}(y) - f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(y)\right]$$

$$= f_{T_1}(x)\overline{F}_{T_0}(x)f_{T_2}(y) + f_{T_0}(x)\overline{F}_{T_1}(x)f_{T_2}(y) = f_X(x)f_{T_2}(y),$$

where according to Proposition 3.3.4,

$$f_X(x) = f_{T_1}(x)\overline{F}_{T_0}(x) + f_{T_0}(x)\overline{F}_{T_1}(x).$$

Similar calculations lead to $f_{X,Y}(x,y)$ in the case $y > x > 0$. In the case $x = y > 0$, the following relation

$$P[\min(T_0,T_1) \in dx, \min(T_0,T_2) \in dx] = P[T_0 \in dx, T_1 > x, T_2 > x],$$

together with the hypothesis that T_0, T_1, T_2 are mutually independent yield the desired result for $f_{X,Y}(x,x)$. □

Proposition 3.3.8. *If (X,Y) follows a BVEr distribution, then*

$$P(X = Y) = \frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!}\sum_{n=0}^{k_1+k_2-2}\frac{c_n(n+k_0-1)!}{(\lambda_1+\lambda_2+\lambda_0)^n},$$

where c_n , $n = 0, 1, ..., k_1 + k_2 - 2$, are given by (3.3.5).

PROOF. Using the result from Proposition 3.3.7, we obtain

$$P(X=Y)=\int_0^\infty f_{X,Y}(x,x)dx=\int_0^\infty f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x)dx$$

$$=\int_0^\infty \frac{\lambda_0^{k_0}x^{k_0-1}}{(k_0-1)!}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_2-1}\frac{\lambda_2^n x^n}{n!}e^{-(\lambda_1+\lambda_2+\lambda_0)x}dx$$

$$=\frac{\lambda_0^{k_0}}{(k_0-1)!}\sum_{n=0}^{k_1+k_2-2}c_n\int_0^\infty x^{n+k_0-1}e^{-(\lambda_1+\lambda_2+\lambda_0)x}dx$$

$$=\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(k_0-1)!}\sum_{n=0}^{k_1+k_2-2}\frac{c_n(n+k_0-1)!}{(\lambda_1+\lambda_2+\lambda_0)^n},$$

where at the last step we used the following result

$$\int_0^\infty y^n e^{-\alpha y}dy=\frac{n!}{\alpha^{n+1}}. \tag{3.3.20}$$

□

Note that the result of Proposition 3.3.8 for the particular case $k_0=k_1=k_2=1$ becomes

$$P(X=Y)=\tfrac{\lambda_0}{\lambda_1+\lambda_2+\lambda_0},$$

a result that was mentioned in Section 1.4 for the BVE distribution.

3.3.4. Conditional probability distribution function and conditional expectation

In this subsection, we derive the conditional probability density function and the conditional expectation for the bivariate exponential distribution.

Proposition 3.3.9. *If (X,Y) follows a BVEr distribution, then the conditional probability density function of X given $Y=y$, $y>0$ has the following form*

$$f_{X|Y}(x|y)=\begin{cases}\frac{f_X(x)f_{T_2}(y)}{f_Y(y)}, & \text{for } x>y>0,\\[2ex] f_{T_1}(x), & \text{for } y>x>0,\\[2ex] \frac{f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x)}{f_Y(x)}, & \text{for } x=y>0,\end{cases}$$

where $f_X(x)$ and $f_Y(y)$ are given by Proposition 3.3.4.

Proof. In the context of Propositions 3.3.4 and 3.3.7, the conditional probability density function of X given $Y=y,\ y>0$ is easily established from the relation

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

□

Corollary 3.3.1. *The conditional expectation of X given $Y=y$ is*

$$E[X|Y=y]=\frac{k_1}{\lambda_1}\left[1-e^{-\lambda_1 y}\sum_{n=0}^{k_1}\frac{(\lambda_1 y)^n}{n!}\right]+\frac{yf_{T_0}(y)\overline{F}_{T_1}(y)\overline{F}_{T_2}(y)}{f_Y(y)}$$

$$+\frac{f_{T_2}(y)e^{-(\lambda_1+\lambda_0)y}}{f_Y(y)}\sum_{i=0}^{1}\frac{\lambda_{1-i}^{k_{1-i}}}{(k_{1-i}-1)!}\sum_{n=0}^{k_i-1}\frac{(n+k_{1-i})!\lambda_i^n}{n!(\lambda_1+\lambda_0)^{n+k_{1-i}+1}}\sum_{l=0}^{n+k_{1-i}}\frac{(\lambda_1+\lambda_0)^l y^l}{l!}.$$

Proof. The result is a consequence of the proposition above, i.e.,

$$E[X|Y=y]=\int_0^\infty xf_{X|Y}(x|y)dx=\int_0^y xf_{T_1}(x)dx+\int_y^\infty x\frac{f_X(x)f_{T_2}(y)}{f_Y(y)}dx$$

$$+\frac{yf_{T_0}(y)\overline{F}_{T_1}(y)\overline{F}_{T_2}(y)}{f_Y(y)}. \tag{3.3.21}$$

Now we compute the integrals

$$I_1=\int_0^y xf_{T_1}(x)dx=\int_0^y x\frac{\lambda_1^{k_1}x^{k_1-1}e^{-\lambda_1 x}}{(k_1-1)!}dx=\frac{\lambda_1^{k_1}}{(k_1-1)!}\int_0^y x^{k_1}e^{-\lambda_1 x}dx$$

$$=\frac{k_1}{\lambda_1}\left[1-e^{-\lambda_1 y}\sum_{n=0}^{k_1}\frac{(\lambda_1 y)^n}{n!}\right],$$

and

$$I_2=\int_y^\infty x\frac{f_X(x)f_{T_2}(y)}{f_Y(y)}dx=\frac{f_{T_2}(y)}{f_Y(y)}\int_y^\infty xf_X(x)dx$$

$$=\frac{f_{T_2}(y)}{f_Y(y)}\frac{\lambda_1^{k_1}}{(k_1-1)!}\sum_{n=0}^{k_0-1}\frac{\lambda_0^n}{n!}\int_y^\infty x^{n+k_1}e^{-(\lambda_1+\lambda_0)x}dx$$

$$+\frac{f_{T_2}(y)}{f_Y(y)}\frac{\lambda_0^{k_0}}{(k_0-1)!}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n}{n!}\int_y^\infty x^{n+k_0}e^{-(\lambda_1+\lambda_0)x}dx$$

$$= \frac{f_{T_2}(y)e^{-(\lambda_1+\lambda_0)y}}{f_Y(y)} \sum_{i=0}^{1} \frac{\lambda_{1-i}^{k_{1-i}}}{(k_{1-i}-1)!} \sum_{n=0}^{k_i-1} \frac{(n+k_{1-i})!\lambda_i^n}{n!(\lambda_1+\lambda_0)^{n+k_{1-i}+1}} \sum_{l=0}^{n+k_{1-i}} \frac{(\lambda_1+\lambda_0)^l y^l}{l!},$$

in view of Proposition 3.3.4 and formula (3.3.7). By replacing both I_1 and I_2 in (3.3.21), the desired result is obtained. □

Similar calculations lead to the conditional probability density function and expectation of Y given $X = x$, which are given by

$$f_{Y|X}(y|x) = \begin{cases} f_{T_2}(y), & \text{for } x > y > 0, \\ \frac{f_Y(y) f_{T_1}(x)}{f_X(x)}, & \text{for } y > x > 0, \\ \frac{f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x)}{f_X(x)}, & \text{for } x = y > 0, \end{cases}$$

and

$$E[Y|X=x] = \frac{k_2}{\lambda_2}\left[1 - e^{-\lambda_2 x}\sum_{n=0}^{k_2}\frac{(\lambda_2 x)^n}{n!}\right] + \frac{x f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x)}{f_X(x)}$$

$$+\frac{f_{T_1}(x)e^{-(\lambda_2+\lambda_0)x}}{f_X(x)} \frac{\lambda_2^{k_2}}{(k_2-1)!} \sum_{n=0}^{k_0-1} \frac{(n+k_2)!\lambda_0^n}{n!(\lambda_2+\lambda_0)^{n+k_2+1}} \sum_{l=0}^{n+k_2} \frac{(\lambda_2+\lambda_0)^l x^l}{l!}$$

$$+\frac{f_{T_1}(x)e^{-(\lambda_2+\lambda_0)x}}{f_X(x)} \frac{\lambda_0^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_2-1} \frac{(n+k_0)!\lambda_2^n}{n!(\lambda_2+\lambda_0)^{n+k_0+1}} \sum_{l=0}^{n+k_0} \frac{(\lambda_2+\lambda_0)^l x^l}{l!},$$

respectively.

3.4. Laplace transform and moments

In this section, we derive the Laplace transform of the bivariate Erlang distribution and moments. We also present these results in a special case where random variables T_1 and T_2 are exponentially distributed.

The Laplace transform of X and Y defined as $L_{X,Y}(r,t) = E[e^{-rX-tY}]$ is described by the following proposition.

Proposition 3.4.1. *The Laplace transform of (X, Y), where (X, Y) follows a BVEr distribution, is given by*

$$L_{X,Y}(r,t) = g(r,t) - g(\infty,t) - g(r,\infty) + 1, \quad r,t > 0, \ \textit{with}$$

$$g(r,t) = \left[1 - \frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}}\right] \sum_{n=0}^{k_1+k_0-2} \frac{a_n r n!}{(\lambda_1+\lambda_0+r)^{n+1}}$$

$$+ \sum_{n=0}^{k_1+k_2+k_0-3} \frac{(te_n - d_n) r n!}{(\lambda_1+\lambda_2+\lambda_0+r+t)^{n+1}},$$

$$g(\infty,t) = \sum_{n=0}^{k_2+k_0-2} \frac{b_n t n!}{(\lambda_2+\lambda_0+t)^{n+1}},$$

$$g(r,\infty) = \sum_{n=0}^{k_1+k_0-2} \frac{a_n r n!}{(\lambda_1+\lambda_0+r)^{n+1}},$$

where d_n and e_n are the coefficients of the following polynomials of degree $k_1 + k_2 + k_0 - 3$:

$$\sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n x^n}{n!} \left[1 - \frac{\lambda_2^{k_2-n}}{(\lambda_2+t)^{k_2-n}}\right] = \sum_{n=0}^{k_1+k_2+k_0-3} d_n x^n, \textit{ and}$$

$$\sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_2+k_0-2} \frac{(\lambda_2+\lambda_0+t)^n x^n}{n!} \sum_{l=n}^{k_2+k_0-2} \frac{b_l l!}{(\lambda_2+\lambda_0+t)^{l+1}} = \sum_{n=0}^{k_1+k_2+k_0-3} e_n x^n.$$

PROOF. The computation of the Laplace transform can be done by using either the joint pdf of (X, Y) and in this case we have that

$$L_{X,Y}(r,t) = \int_0^\infty \int_0^x e^{-rx-ty} f_{X,Y}(x,y)\,dy\,dx$$

$$+ \int_0^\infty \int_0^y e^{-rx-ty} f_{X,Y}(x,y)\,dx\,dy + \int_0^\infty e^{-(r+t)x} f_{X,Y}(x,x)\,dx,$$

or the result of Young (1917) on integration by parts, namely if $G(x,y)$ is of bounded variation on finite intervals and $G(x,0) = 0 = G(0,y)$, then

$$\int_0^\infty \int_0^\infty G(x,y)\,dF_{X,Y}(x,y) = \int_0^\infty \int_0^\infty \overline{F}_{X,Y}(x,y)\,dG(x,y). \qquad (3.4.1)$$

We adopt the second method and motivated by Marshall and Olkin (1967a), let $G(x,y) = (1-e^{-rx})(1-e^{-ty})$. Then, the Laplace transform of (X,Y) is computed from the relation

$$L_{X,Y}(r,t) = g(r,t) - g(\infty,t) - g(r,\infty) + 1,$$

where we denote by $g(r,t) = \int_0^\infty \int_0^\infty (1-e^{-rx})(1-e^{-ty})dF_{X,Y}(x,y)$.
According to (3.4.1), we have that
$g(r,t) = \int_0^\infty \int_0^\infty \overline{F}_{X,Y}(x,y) rte^{-rx-ty} dxdy$

$$= rt \int_0^\infty \int_0^x e^{-(\lambda_1+\lambda_0+r)x-(\lambda_2+t)y} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n y^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} dydx$$

$$+rt \int_0^\infty \int_x^\infty e^{-(\lambda_1+r)x-(\lambda_2+\lambda_0+t)y} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n y^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n y^n}{n!} dydx. \tag{3.4.2}$$

We denote the two terms in (3.4.2) by I_3 and I_4, respectively. Therefore,

$$I_3 = rt \int_0^\infty e^{-(\lambda_1+\lambda_0+r)x} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n}{n!} \int_0^x e^{-(\lambda_2+t)y} y^n dydx$$

$$= rt \sum_{n=0}^{k_2-1} \frac{\lambda_2^n}{(\lambda_2+t)^{n+1}} \int_0^\infty e^{-(\lambda_1+\lambda_0+r)x} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} dx$$

$$-rt \int_0^\infty e^{-(\lambda_1+\lambda_0+\lambda_2+r+t)x} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n}{(\lambda_2+t)^{n+1}} \sum_{l=0}^{n} \frac{(\lambda_2+t)^l x^l}{l!} dx, \tag{3.4.3}$$

where the following result has been used

$$\int_0^x y^n e^{-\alpha y} dy = \frac{n!}{\alpha^{n+1}} \left[1 - e^{-\alpha x} \sum_{l=0}^{n} \frac{\alpha^l x^l}{l!} \right].$$

Further, computing the following sums from (3.4.3) as

$$\sum_{n=0}^{k_2-1} \frac{\lambda_2^n}{(\lambda_2+t)^{n+1}} = \frac{1}{t} \left[1 - \frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}} \right] \text{ and}$$

$$\sum_{n=0}^{k_2-1} \frac{\lambda_2^n}{(\lambda_2+t)^{n+1}} \sum_{l=0}^{n} \frac{(\lambda_2+t)^l x^l}{l!} = \sum_{n=0}^{k_2-1} \frac{\lambda_2^n x^n}{n!t} \left[1 - \frac{\lambda_2^{k_2-n}}{(\lambda_2+t)^{k_2-n}} \right],$$

the right-hand side of (3.4.3) is equal to

$$= r \left[1 - \frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}} \right] \int_0^\infty e^{-(\lambda_1+\lambda_0+r)x} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} dx$$

$$-rt \int_0^\infty e^{-(\lambda_1+\lambda_0+\lambda_2+r+t)x} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n x^n}{n!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n x^n}{n!t} \left[1 - \frac{\lambda_2^{k_2-n}}{(\lambda_2+t)^{k_2-n}} \right] dx$$

$$= r\left[1-\frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}}\right]\sum_{n=0}^{k_1+k_0-2} a_n \int_0^{\infty} e^{-(\lambda_1+\lambda_0+r)x}x^n dx$$

$$-r\sum_{n=0}^{k_1+k_2+k_0-3} d_n \int_0^{\infty} e^{-(\lambda_1+\lambda_0+\lambda_2+r+t)x}x^n dx$$

$$= \left[1-\frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}}\right]\sum_{n=0}^{k_1+k_0-2}\frac{a_n r n!}{(\lambda_1+\lambda_0+r)^{n+1}}$$

$$-\sum_{n=0}^{k_1+k_2+k_0-3}\frac{d_n r n!}{(\lambda_1+\lambda_0+\lambda_2+r+t)^{n+1}},$$

where a_n, $n = 0, 1, ..., k_1 + k_0 - 2$ are given by (3.3.3) and d_n, $n = 0, 1, ..., k_1 + k_2 + k_0 - 3$ are the coefficients of the polynomial:

$$\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_0-1}\frac{\lambda_0^n x^n}{n!}\sum_{n=0}^{k_2-1}\frac{\lambda_2^n x^n}{n!}\left[1-\frac{\lambda_2^{k_2-n}}{(\lambda_2+t)^{k_2-n}}\right] = \sum_{n=0}^{k_1+k_2+k_0-3} d_n x^n.$$

Now for I_4 we have

$$I_4 = rt\int_0^{\infty}\int_x^{\infty} e^{-(\lambda_1+r)x-(\lambda_2+\lambda_0+t)y}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_2-1}\frac{\lambda_2^n y^n}{n!}\sum_{n=0}^{k_0-1}\frac{\lambda_0^n y^n}{n!}dydx$$

$$= rt\int_0^{\infty} e^{-(\lambda_1+r)x}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_2+k_0-2} b_n\int_x^{\infty} e^{-(\lambda_2+\lambda_0+t)y}y^n dydx$$

$$= rt\int_0^{\infty} e^{-(\lambda_1+r)x}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_2+k_0-2}\frac{b_n n! e^{-(\lambda_2+\lambda_0+t)x}}{(\lambda_2+\lambda_0+t)^{n+1}}\sum_{l=0}^{n}\frac{(\lambda_2+\lambda_0+t)^l x^l}{l!}dx$$

$$= \int_0^{\infty} e^{-(\lambda_1+\lambda_2+\lambda_0+r+t)x}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_2+k_0-2}\frac{(\lambda_2+\lambda_0+t)^n x^n}{n!}\sum_{l=n}^{k_2+k_0-2}\frac{b_l l! rtdx}{(\lambda_2+\lambda_0+t)^{l+1}}$$

$$= rt\sum_{n=0}^{k_1+k_2+k_0-3} e_n\int_0^{\infty} e^{-(\lambda_1+\lambda_2+\lambda_0+r+t)x}x^n dx = \sum_{n=0}^{k_1+k_2+k_0-3}\frac{e_n n! rt}{(\lambda_1+\lambda_2+\lambda_0+r+t)^{n+1}},$$

where e_n $(n = 0, 1, ..., k_1 + k_2 + k_0 - 3)$ is defined by the following polynomial:

$$\sum_{n=0}^{k_1-1}\frac{\lambda_1^n x^n}{n!}\sum_{n=0}^{k_2+k_0-2}\frac{(\lambda_2+\lambda_0+t)^n x^n}{n!}\sum_{l=n}^{k_2+k_0-2}\frac{b_l l!}{(\lambda_2+\lambda_0+t)^{l+1}} = \sum_{n=0}^{k_1+k_2+k_0-3} e_n x^n.$$

Substituting I_3 and I_4 in (3.4.2) yields the desired result for $g(r,t)$. To complete the proof, we compute $g(\infty,t)$ and $g(r,\infty)$ as follows:

$$g(\infty,t) = a_0\left[1-\frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}}\right] - d_0 + te_0$$

$$= \left[1-\frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}}\right] - \left[1-\frac{\lambda_2^{k_2}}{(\lambda_2+t)^{k_2}}\right] + \sum_{l=0}^{k_2+k_0-2}\frac{tb_l l!}{(\lambda_2+\lambda_0+t)^{l+1}}$$

$$= \sum_{n=0}^{k_2+k_0-2}\frac{b_n tn!}{(\lambda_2+\lambda_0+t)^{n+1}},$$

$$\text{and}\quad g(r,\infty) = \sum_{n=0}^{k_1+k_0-2}\frac{a_n rn!}{(\lambda_1+\lambda_0+r)^{n+1}}.$$

□

In the following, we give the expression of the Laplace transform established in Proposition 3.4.1 considering a particular case.

Corollary 3.4.1. *If T_1 and T_2 are exponentially distributed with the corresponding parameters λ_1 and λ_2, and T_0 follows an Erlang $(k_0,\ \lambda_0)$ distribution, then the Laplace transform of (X,Y) is given by*

$$L_{X,Y}(r,t) = \frac{1}{(\lambda_1+r)(\lambda_2+t)}\left[\frac{\lambda_0^{k_0}rt}{(\lambda_1+\lambda_2+\lambda_0+r+t)^{k_0}} + \frac{\lambda_0^{k_0}\lambda_2 r}{(\lambda_1+\lambda_0+r)^{k_0}} + \frac{\lambda_0^{k_0}\lambda_1 t}{(\lambda_2+\lambda_0+t)^{k_0}} + \lambda_1\lambda_2\right].$$

Proof. This particular case implies that $k_1 = k_2 = 1$ and by Proposition 3.4.1, we obtain

$$g(r,t) = \frac{\lambda_0^{k_0}rt}{(\lambda_1+\lambda_2+\lambda_0+r+t)^{k_0}(\lambda_1+r)(\lambda_2+t)} - \frac{\lambda_0^{k_0}rt}{(\lambda_1+\lambda_0+r)^{k_0}(\lambda_1+r)(\lambda_2+t)}$$

$$-\frac{\lambda_0^{k_0}rt}{(\lambda_2+\lambda_0+t)^{k_0}(\lambda_2+t)(\lambda_1+r)} + \frac{rt}{(\lambda_1+r)(\lambda_2+t)},$$

$$g(\infty,t) = -\frac{\lambda_0^{k_0}t}{(\lambda_2+\lambda_0+t)^{k_0}(\lambda_2+t)} + \frac{t}{(\lambda_2+t)},$$

$$\text{and } g(r,\infty) = -\frac{\lambda_0^{k_0}r}{(\lambda_1+\lambda_0+r)^{k_0}(\lambda_1+r)} + \frac{r}{(\lambda_1+r)}.$$

Then the first relation in Proposition 3.4.1 leads to the result of the corollary. □

If, in Corollary 3.4.1, we further consider that T_0 is exponentially distributed with parameter λ_0, the Laplace transform is

$$L_{X,Y}(r,t) = \frac{1}{(\lambda_1+r)(\lambda_2+t)}\left[\frac{\lambda_0 rt}{\lambda_1+\lambda_2+\lambda_0+r+t} + \frac{\lambda_0\lambda_2 r}{\lambda_1+\lambda_0+r} + \frac{\lambda_0\lambda_1 t}{\lambda_2+\lambda_0+t} + \lambda_1\lambda_2\right]$$

$$= \frac{(\lambda_1+\lambda_2+\lambda_0+r+t)(\lambda_1+\lambda_0)(\lambda_2+\lambda_0)+rt\lambda_0}{(\lambda_1+\lambda_2+\lambda_0+r+t)(\lambda_1+\lambda_0+r)(\lambda_2+\lambda_0+t)},$$

which represents the Laplace transform of the BVE distribution, as property (4) of Proposition 1.4.2 illustrated.

Remark 3.4.1. *By using Proposition* 3.4.1*, we compute* $E(XY)$ *as*

$$E(XY) = \frac{\partial^2 L_{X,Y}(r,t)}{\partial r \partial t}|_{r=t=0} = \lim_{r,t\to 0} \frac{g(r,t)}{rt}.$$

Proposition 3.4.2. *The* r*-th moments* $(r = 1, 2, ...)$ *of* X *and* Y *are given by*

$$E(X^r) = \frac{\lambda_1^{k_1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n (n+k_1+r-1)!}{(\lambda_1+\lambda_0)^{n+k_1+r} n!} + \frac{\lambda_0^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n (n+k_0+r-1)!}{(\lambda_1+\lambda_0)^{n+k_0+r} n!},$$

$$E(Y^r) = \frac{\lambda_2^{k_2}}{(k_2-1)!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n (n+k_2+r-1)!}{(\lambda_2+\lambda_0)^{n+k_2+r} n!} + \frac{\lambda_0^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_2-1} \frac{\lambda_2^n (n+k_0+r-1)!}{(\lambda_2+\lambda_0)^{n+k_0+r} n!}.$$

Proof. Using the probability density function of X from Proposition 3.3.4, for $r = 1, 2, ...,$ we have

$$E(X^r) = \int_0^\infty x^r f_X(x) dx$$

$$= \frac{\lambda_1^{k_1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n}{n!} \int_0^\infty x^{n+k_1+r-1} e^{-(\lambda_1+\lambda_0)x} dx + \frac{\lambda_0^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n}{n!} \int_0^\infty x^{n+k_0+r-1} e^{-(\lambda_1+\lambda_0)x} dx$$

$$= \frac{\lambda_1^{k_1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{\lambda_0^n (n+k_1+r-1)!}{(\lambda_1+\lambda_0)^{n+k_1+r} n!} + \frac{\lambda_0^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{\lambda_1^n (n+k_0+r-1)!}{(\lambda_1+\lambda_0)^{n+k_0+r} n!},$$

where formula (3.3.20) was applied. The r-th moment of Y is computed in a similar way. □

Corollary 3.4.2. *If* T_1 *and* T_2 *are exponentially distributed with the corresponding parameters* λ_1 *and* λ_2 *and* T_0 *follows an Erlang (*k_0*,* λ_0*) distribution, then the covariance and correlation structure are given by*

$$Cov(X,Y) = \frac{1}{\lambda_1 \lambda_2} \left[\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}} - \frac{\lambda_0^{2k_0}}{(\lambda_1+\lambda_0)^{k_0} (\lambda_2+\lambda_0)^{k_0}} \right], \tag{3.4.4}$$

and $$Corr(X,Y) = \frac{\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}} - \frac{\lambda_0^{2k_0}}{(\lambda_1+\lambda_0)^{k_0}(\lambda_2+\lambda_0)^{k_0}}}{\sqrt{\prod_{i=1}^{2} \left[1 - \frac{2k_0 \lambda_i \lambda_0^{k_0}}{(\lambda_i+\lambda_0)^{k_0+1}} - \frac{\lambda_0^{2k_0}}{(\lambda_i+\lambda_0)^{2k_0}} \right]}}. \tag{3.4.5}$$

The correlation between X *and* Y *is non-negative, that is,* $0 \le Corr(X,Y) \le 1$.

PROOF. From both Corollary 3.4.1 and Remark 3.4.1, we obtain

$$E(XY) = \frac{\partial^2 L_{X,Y}(r,t)}{\partial r \partial t}|_{r=t=0} = \lim_{r,t\to 0} \frac{g(r,t)}{rt}$$

$$= \frac{1}{\lambda_1\lambda_2}\left[\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}} - \frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_0)^{k_0}} - \frac{\lambda_0^{k_0}}{(\lambda_2+\lambda_0)^{k_0}} + 1\right].$$

Also, for $k_1 = k_2 = 1$, Proposition 3.4.2 gives

$$E(X) = \frac{1}{\lambda_1}\left[1 - \frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_0)^{k_0}}\right],$$

$$E(X^2) = \frac{2}{\lambda_1^2}\left[1 - \frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_0)^{k_0}} - \frac{k_0\lambda_1\lambda_0^{k_0}}{(\lambda_1+\lambda_0)^{k_0+1}}\right],$$

and similar results for Y. The proof is completed by using the formulas

$$Cov(X,Y) = E(XY) - E(X)E(Y) \text{ and } Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}.$$

Clearly,

$$\frac{\lambda_0^{k_0}}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}} - \frac{\lambda_0^{2k_0}}{(\lambda_1+\lambda_0)^{k_0}(\lambda_2+\lambda_0)^{k_0}}$$

$$= \frac{\lambda_0^{k_0}\left[(\lambda_1\lambda_2+\lambda_1\lambda_0+\lambda_2\lambda_0+\lambda_0^2)^{k_0} - (\lambda_1\lambda_0+\lambda_2\lambda_0+\lambda_0^2)^{k_0}\right]}{(\lambda_1+\lambda_2+\lambda_0)^{k_0}(\lambda_1+\lambda_0)^{k_0}(\lambda_2+\lambda_0)^{k_0}} \geq 0,$$

and hence, $Corr(X,Y) \in [0,1]$. □

If in Corollary 3.4.2, we further consider that T_0 is exponentially distributed with parameter λ_0, then (3.4.4) and (3.4.5) become the results described by property (5) of Proposition 1.4.2, for the BVE distribution.

An upper bound for the joint survival function and a lower bound for the joint distribution function of a BVEr distribution are illustrated below.

Proposition 3.4.3. *If (X,Y) follows a BVEr distribution, then*

$$P[X>\alpha, Y>\beta] \leq \frac{E(X)+E(Y)}{\alpha+\beta} \textit{ and } P[X\leq\alpha, Y\leq\beta] \geq 1 - \frac{E(X)}{\alpha} - \frac{E(Y)}{\beta},$$

where

$$E(X) = \frac{\lambda_1^{k_1}}{(k_1-1)!}\sum_{n=0}^{k_0-1}\frac{\lambda_0^n(n+k_1)!}{(\lambda_1+\lambda_0)^{n+k_1+1}n!} + \frac{\lambda_0^{k_0}}{(k_0-1)!}\sum_{n=0}^{k_1-1}\frac{\lambda_1^n(n+k_0)!}{(\lambda_1+\lambda_0)^{n+k_0+1}n!},$$

$$E(Y) = \frac{\lambda_2^{k_2}}{(k_2-1)!}\sum_{n=0}^{k_0-1}\frac{\lambda_0^n(n+k_2)!}{(\lambda_2+\lambda_0)^{n+k_2+1}n!} + \frac{\lambda_0^{k_0}}{(k_0-1)!}\sum_{n=0}^{k_2-1}\frac{\lambda_2^n(n+k_0)!}{(\lambda_2+\lambda_0)^{n+k_0+1}n!},$$

and α, β are positive constants.

PROOF. The proof is completed by using Markov's inequality:

$$P[X > \alpha] \leq \tfrac{E(X)}{\alpha},$$

and the following inequalities

$$P[X > \alpha, Y > \beta] \leq P[X + Y > \alpha + \beta],$$

and

$$P[X \leq \alpha, Y \leq \beta] = 1 - P[\{X > \alpha\} \cup \{Y > \beta\}] \geq 1 - P[X > \alpha] - P[Y > \beta].$$

□

3.5. Mixture of BVEr distributions

In this section, we propose an extension of the BVEr model to a bivariate model which assumes that for each $i = 0, 1, 2$, the random variable T_i is described by a mixture of Erlang distributions with common rate parameter λ_i. Therefore, the survival function and probability density function of T_i, for $i = 0, 1, 2$, and $t \geq 0$ are given by

$$\overline{F}_{T_i}(t) = \sum_{j=1}^{N_i} \sum_{n=0}^{k_{ij}-1} \alpha_{ij} e^{-\lambda_i t} \frac{\lambda_i^n t^n}{n!}, \tag{3.5.1}$$

$$f_{T_i}(t) = \sum_{j=1}^{N_i} \alpha_{ij} e^{-\lambda_i t} \frac{\lambda_i^{k_{ij}} t^{k_{ij}-1}}{(k_{ij}-1)!}, \tag{3.5.2}$$

where $k_{i1}, k_{i2}, ..., k_{iN_i}$ are the shape parameters of the Erlang distributions, and $\alpha_{i1}, .., \alpha_{iN_i}$ are nonnegative weights and sum to 1.

In this case, the joint survival function of $X = \min(T_1, T_0)$ and $Y = \min(T_2, T_0)$ becomes

$$\begin{aligned}\overline{F}_{X,Y}(x, y) &= \overline{F}_{T_1}(x)\overline{F}_{T_2}(y)\overline{F}_{T_0}(\max(x, y)) \\ &= e^{-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x,y)} \sum_{j=1}^{N_1} \sum_{n=0}^{k_{1j}-1} \alpha_{1j} \frac{(\lambda_1 x)^n}{n!} \\ &\times \sum_{j=1}^{N_2} \sum_{n=0}^{k_{2j}-1} \alpha_{2j} \frac{(\lambda_2 y)^n}{n!} \sum_{j=1}^{N_0} \sum_{n=0}^{k_{0j}-1} \alpha_{0j} \frac{(\lambda_0 \max(x, y))^n}{n!},\end{aligned} \tag{3.5.3}$$

and the random vector $(X,\ Y)$ is said to follow a bivariate mixture of Erlang distributions (BVMEr) of Marshall-Olkin type.

After expanding (3.5.3), we find that the survival function of (X, Y) can be written as the survival function of a mixture of the bivariate Erlang distributions introduced in Section 3.3. Therefore,

$$\overline{F}_{X,Y}(x, y) = \sum_{j=1}^{N_1} \sum_{l=1}^{N_2} \sum_{m=1}^{N_3} \beta_{jlm} \overline{F}_{jlm}(x, y), \tag{3.5.4}$$

where $\overline{F}_{jlm}(x, y)$ is the survival function of the BVEr distribution with parameters $(k_{1j},\ \lambda_1\ ,\ k_{2l},\ \lambda_2\ ,\ k_{3m},\ \lambda_3)$, and the corresponding β_{jlm} is of the form $\alpha_{1j}\alpha_{2l}\alpha_{3m}$, with $j \in \{1, 2, ..., N_1\}$, $l \in \{1, 2, ..., N_2\}$, $m \in \{1, 2, ..., N_3\}$. Clearly, $\beta_{jlm} \geq 0$ and $\sum\limits_{j=1}^{N_1} \sum\limits_{l=1}^{N_2} \sum\limits_{m=1}^{N_3} \beta_{jlm} = 1$.

If we further adopt the notation $L_{jlm}(r, t)$, $j = 1, 2, ..., N_1$, $l = 1, 2, ..., N_2$, $m = 1, 2, ..., N_3$, for the Laplace transform of the the BVEr distributions defined by the survival function $\overline{F}_{jlm}(x, y)$, then the Laplace transform for the BVMEr distribution will be of the form

$$L_{X,Y}(r, t) = \sum_{j=1}^{N_1} \sum_{l=1}^{N_2} \sum_{m=1}^{N_3} \beta_{jlm} L_{jlm}(r, t),$$

in virtue of the relation (3.5.4).

The same procedure used in Section 3.3 yields the marginal survival and probability density functions for the BVMEr distribution, that is,

$$\overline{F}_X(x) = P(X > x) = \overline{F}_{T_1}(x)\overline{F}_{T_0}(x),\ \overline{F}_Y(y) = P(Y > y) = \overline{F}_{T_2}(y)\overline{F}_{T_0}(y),$$

$$f_X(x) = -\frac{d}{dx}\overline{F}_X(x) = f_{T_1}(x)\overline{F}_{T_0}(x) + f_{T_0}(x)\overline{F}_{T_1}(x) \text{ and}$$

$$f_Y(y) = -\frac{d}{dy}\overline{F}_Y(y) = f_{T_2}(y)\overline{F}_{T_0}(y) + f_{T_0}(y)\overline{F}_{T_2}(y),$$

where $\overline{F}_{T_i}(t)$ and $f_{T_i}(t)$ $(i = 0, 1, 2)$ are given by (3.5.1) and (3.5.2), respectively.

Proposition 3.5.1. *The joint probability density function of the BVMEr distribution has the following form*

$$f_{X,Y}(x, y) = \begin{cases} \left[f_{T_1}(x)\overline{F}_{T_0}(x) + f_{T_0}(x)\overline{F}_{T_1}(x)\right] f_{T_2}(y), & \textit{for } x > y > 0, \\ \left[f_{T_2}(y)\overline{F}_{T_0}(y) + f_{T_0}(y)\overline{F}_{T_2}(y)\right] f_{T_1}(x), & \textit{for } y > x > 0, \\ f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x), & \textit{for } x = y > 0. \end{cases}$$

Proof. The proof is the same as that of Proposition 3.3.7, with the only difference that $\overline{F}_{T_i}(t)$ and $f_{T_i}(t)$ $(i = 0, 1, 2)$ are given by (3.5.1) and (3.5.2), respectively. □

3.6. Interpretations and possible applications in insurance and finance

The bivariate distributions proposed in this thesis have natural interpretations and for example, they can be applied in fatal shock models or in competing risks models.

For the fatal shock model we have the following interpretation. Suppose that two components labeled 1 and 2 in a system are subject to three types of events called shocks in such a way that if the first (second) type of shock occurs, component 1 (2) fails, whereas at the occurrence of the third type of shock, both 1 and 2 fail. Assume that the occurrences of these shocks are governed by three independent renewal processes with the corresponding inter-arrival times denoted by T_1, T_2 and T_0. The lifetime of the first component is the random variable $X = \min(T_1, T_0)$ and that of the second component is $Y = \min(T_2, T_0)$.

Our bivariate Erlang distribution can be used in modeling the random vector (X, Y) under the assumptions that the shocks arrive as independent Erlang processes. In the special case of the bivariate exponential distribution proposed by Marshall and Olkin (1967a), shocks arrive as independent Poisson processes with intensity parameters λ_1, λ_2 and λ_0, respectively.

In the context of the competing risks model, consider a system with two components, say 1 and 2, having the survival times denoted by X and Y, respectively. It is assumed that there are three different causes of failures, which may affect the system. Due to cause 1, only component 1 can fail and similarly, due to cause 2 only component 2 can fail. But due to cause 3, both the components fail at the same time. If the lifetime distributions of the different causes are Erlang or finite mixture of Erlang distributions, and they are independent, then the lifetime of the two-component system follows a BVEr or a mixture of BVEr distributions, respectively.

Due to these interpretations, the distributions we propose in this thesis have potential applications in various fields such as the actuarial theory of life insurance, finance, reliability theory and medical studies.

In the context of life insurance, annuities are contractual guarantees that provide periodic income over the lifetime of an individual, called annuitant. Classical models of dependent lives are called "common shock" models [Marshall and Olkin (1988); Bowers et al. (1997)]. This model is used in modeling the dependency between two lives in one life insurance contract, in the sense that the lifetimes of two persons, say T_1 and T_2, are independent unless a common shock causes the death of both. For example, a contagious deadly disease, a natural catastrophe or a car accident may affect the lives of the two spouses. Thus, if T_0 denotes the time until the common disaster, the actual ages-at-death are represented by

$$X = \min(T_1, T_0) \text{ and } Y = \min(T_2, T_0). \tag{3.6.1}$$

As an example, Frees, Carriere, and Valdez (1996) considered a common shock model to estimate the joint survival probability of a couple and to investigate the annuity contracts valuation.

It follows from (3.6.1) that if the lifetimes T_1, T_2 and T_0 follow Erlang distributions, then the bivariate Erlang distribution proposed in this thesis can be a potential model for life insurance modeling.

Provided that a set of lives is viewed as a status, the random variable $Z = \min(X, Y)$ represents the time-until-failure of a joint-life status, which fails when the first death occurs, while the random variable $W = \max(X, Y)$ is the time-until-failure of a last-survivor status, which fails upon the last death.

Therefore, in the case when (X, Y) follows the bivariate Erlang distribution, the lifetimes X, Y, $\min(X, Y)$ are shown to be finite mixtures of Erlang distributions, while $\max(X, Y)$ is modeled by a weighted average of Erlangs.

The common shock model is easy to implement and it is convenient for annuity and insurance calculations due to its particularly simple form.

This kind of construction is very familiar in the reliability literature where the failure of different kinds of system components is modelled as being contingent on independent shocks that may affect one or more components [see, for example, Barlow and Proschan (1975)]. If X and Y are life lengths of components subject to shocks, then (X, Y) denotes the life length of a two-component system.

A typical probability model popular in reliability and biostatistics is a combination of increasing failure rate and decreasing failure rate. The reason for using this model is that many data sets available in reliability theory, clinical trials, and biostatistics surveys reveal the so-called "bath-tub" shape failure rate. An interpretation of this shape is that the failure rate is initially decreasing during the "infant mortality" phase, then remains relatively constant during the "useful" life phase, and finally reaches the "wear-out" phase, that is, the failure rate increases. The mixture of Erlang distributions is a typical example, as we discussed in Section 1.3 and this can be a reason for considering the bivariate Erlang distribution as a model for the life length of the two-component system (X, Y).

Reliability properties of Erlang mixtures involving the failure rate are discussed by Esary, Marshall and Proschan (1973).

The common shock model can also be applied in finance, where the shocks such as local or global recessions affecting one or more credit-risky assets at a time may cause joint defaults and in this case, the random variables T_i, $i = 0, 1, 2$ are interpreted as arrival times of the shocks while X and Y define the default times. For example, Giesecke (2003), Lindskog and McNeil (2003) used the model where the shocks arrive as independent Poisson processes in the context of credit risk modeling and insurance loss modeling.

The competing risks model arises in actuarial science as well as in survival analysis, systems reliability, and medical studies. The subject of competing risks is called multiple decrement theory in actuarial science [see, Bowers et al. 1997, Chapters 10 and 11]. For example, a person may die because of one of several possible causes: cancer, heart disease, accident, and so on.

As we discussed in Subsection 3.3.1, the BVEr distribution is made up of a singular distribution and a continuous one, while the marginal distributions are continuous. In the univariate case, distributions with singularities are rarely popular but in the multivariate one they can be easily motivated. Indeed, this model can be applied in situations where there exists positive probability of simultaneous failure of Erlang type for two components in a system. For example, the simultaneous failure can be the failure of a two-engine plane because one engine explodes and the adjacent engine is destroyed by the explosion or the failure of paired organs such as ears, eyes, and kidneys (say industrial accidents involving the loss of sight or hearing).

We conclude that the bivariate models introduced in this thesis are plausible models in many practical contexts.

3.7. Inference for the BVEr model

In this section, we focus on the estimation of the parameters for the bivariate Erlang distribution with the joint probability density function given by Proposition 3.3.7. Unfortunately, statistical inference for the BVEr is not an easy task due to the complicated nature of its density function.

Due to the constraints on the shape parameters k_0, k_1, and k_2 to be positive integers, we restrict our investigation to estimating the parameters λ_0, λ_1, and λ_2 in the case where k_0, k_1, and k_2 are assumed known.

3.7.1. Parameter estimation method

In the sequel, we adopt the Expectation-Maximization (EM) algorithm proposed by Karlis (2003) for computing the maximum likelihood (ML) estimators for the Marshall-Olkin bivariate exponential distribution which, as stated earlier, can be obtained from the BVEr distribution by taking $k_0 = k_1 = k_2 = 1$.

The EM method was described and analyzed by Dempster, Laird, and Rubin (1977) and alternatives of the EM algorithm illustrated through examples are discussed by McLachlan and Krishnan (1997).

The trivariate reduction technique used to construct the BVEr distribution involves a random sample that consists of two components, one observed and one unobserved or missing. Therefore, observations on (X, Y) constitute the observed data denoted by (x_i, y_i), $i = 1, 2, .., n$, while the random variables $T_0, T_1,$ and T_2 yield the corresponding unobserved data.

The EM algorithm starts with some initial guess of the parameters as $\Lambda^{(0)} = (\lambda_0^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)})$ and then the successive parameter estimates, $\Lambda^{(k)} = (\lambda_0^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})$, $k = 1, 2, ...$, are generated iteratively by applying the following two alternating steps:

• *E-step*: Compute the conditional expectations with respect to the distribution for the unobserved variables given the observed data (x_i, y_i), for $i = 1, 2, .., n$, and the current parameter estimates $\Lambda^{(k)} = (\lambda_0^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})$ after the k-th iteration. Thus, for $i = 1, 2, .., n$, the following values, usually called pseudo-values, are obtained:

$$a_i = \mathrm{E}(T_{0i} \mid x_i, y_i, \Lambda^{(k)}),\ b_i = \mathrm{E}(T_{1i} \mid x_i, y_i, \Lambda^{(k)}),\ c_i = \mathrm{E}(T_{2i} \mid x_i, y_i, \Lambda^{(k)}).$$

• *M-step*: Re-estimate the parameters to be those with maximum likelihood for a sample from Erlang distributions, using the pseudo-values of the E-step, i.e.,

$$\lambda_0^{(k+1)} = \frac{nk_0}{\sum\limits_{i=1}^{n} a_i},\ \lambda_1^{(k+1)} = \frac{nk_1}{\sum\limits_{i=1}^{n} b_i},\ \lambda_2^{(k+1)} = \frac{nk_2}{\sum\limits_{i=1}^{n} c_i}.$$

The steps are iterated until some convergence criterion is fulfilled, for example, when the estimates of the parameters are no longer changing appreciably.

To complete the description of this parameter estimation method, we need to compute the conditional expectations mentioned in the E-step. For simplicity, we denote the given data by (x, y) and the parameter vector by $\Lambda = (\lambda_0, \lambda_1, \lambda_2)$. The following three cases are considered.

Case 1. Assume $x > y > 0$. It follows that $T_0 \geq x$ and the conditional density of T_0 is described by

$$f_{T_0|X,Y,\Lambda}(t \mid x, y, \Lambda) = \begin{cases} \frac{f_{T_2}(y) f_{T_0}(x) \overline{F}_{T_1}(x)}{f_{X,Y}(x,y)} = \frac{f_{T_0}(x)\overline{F}_{T_1}(x)}{f_X(x)}, & t = x \\ \frac{f_{T_2}(y) f_{T_0}(t) f_{T_1}(x)}{f_{X,Y}(x,y)} = \frac{f_{T_0}(t) f_{T_1}(x)}{f_X(x)}, & t > x, \end{cases}$$

since $f_{X,Y}(x, y) = f_X(x) f_{T_2}(y)$. Then the conditional expectation of T_0 is

$$\mathrm{E}(T_0 \mid x, y, \Lambda) = x f_{T_0|X,Y,\Lambda}(x \mid x, y, \Lambda) + \int_x^\infty t f_{T_0|X,Y,\Lambda}(t \mid x, y, \Lambda) dt$$

$$= \frac{x f_{T_0}(x) \overline{F}_{T_1}(x) + f_{T_1}(x) \frac{k_0}{\lambda_0} e^{-\lambda_0 x} \sum\limits_{n=0}^{k_0} \frac{(\lambda_0 x)^n}{n!}}{f_X(x)}$$

$$= \frac{\frac{\lambda_0^{k_0} x^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{(\lambda_1 x)^n}{n!} + \frac{\lambda_1^{k_1} x^{k_1-1}}{(k_1-1)!} \frac{k_0}{\lambda_0} \sum_{n=0}^{k_0} \frac{(\lambda_0 x)^n}{n!}}{\frac{\lambda_0^{k_0} x^{k_0-1}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{(\lambda_1 x)^n}{n!} + \frac{\lambda_1^{k_1} x^{k_1-1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 x)^n}{n!}}.$$

It is obvious that $\mathrm{E}(T_2 \mid x, y, \Lambda) = y$ and $T_1 \geq x$. Thus, the conditional density of T_1 is given by

$$f_{T_1|X,Y,\Lambda}(t \mid x, y, \Lambda) = \begin{cases} \frac{f_{T_2}(y) f_{T_1}(x) \overline{F}_{T_0}(x)}{f_{X,Y}(x,y)} = \frac{f_{T_1}(x) \overline{F}_{T_0}(x)}{f_X(x)}, & t = x \\ \frac{f_{T_2}(y) f_{T_1}(t) f_{T_0}(x)}{f_{X,Y}(x,y)} = \frac{f_{T_1}(t) f_{T_0}(x)}{f_X(x)}, & t > x, \end{cases}$$

and the conditional expectation is computed as

$$\mathrm{E}(T_1 \mid x, y, \Lambda) = x f_{T_1|X,Y,\Lambda}(x \mid x, y, \Lambda) + \int_x^\infty t f_{T_1|X,Y,\Lambda}(t \mid x, y, \Lambda) dt$$

$$= \frac{x f_{T_1}(x) \overline{F}_{T_0}(x) + f_{T_0}(x) \frac{k_1}{\lambda_1} e^{-\lambda_1 x} \sum_{n=0}^{k_1} \frac{(\lambda_1 x)^n}{n!}}{f_X(x)}$$

$$= \frac{\frac{\lambda_1^{k_1} x^{k_1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 x)^n}{n!} + \frac{\lambda_0^{k_0} x^{k_0-1}}{(k_0-1)!} \frac{k_1}{\lambda_1} \sum_{n=0}^{k_1} \frac{(\lambda_1 x)^n}{n!}}{\frac{\lambda_1^{k_1} x^{k_1-1}}{(k_1-1)!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 x)^n}{n!} + \frac{\lambda_0^{k_0} x^{k_0-1}}{(k_0-1)!} \sum_{n=0}^{k_1-1} \frac{(\lambda_1 x)^n}{n!}}.$$

Case 2. Assume $y > x > 0$. Proceeding in the same manner as in Case 1, we have the following results for the conditional expectations:

$$\mathrm{E}(T_0 \mid x, y, \Lambda) = \frac{y f_{T_0}(y) \overline{F}_{T_2}(y) + f_{T_2}(y) \frac{k_0}{\lambda_0} e^{-\lambda_0 y} \sum_{n=0}^{k_0} \frac{(\lambda_0 y)^n}{n!}}{f_Y(y)}$$

$$= \frac{\frac{\lambda_0^{k_0} y^{k_0}}{(k_0-1)!} \sum_{n=0}^{k_2-1} \frac{(\lambda_2 y)^n}{n!} + \frac{\lambda_2^{k_2} y^{k_2-1}}{(k_2-1)!} \frac{k_0}{\lambda_0} \sum_{n=0}^{k_0} \frac{(\lambda_0 y)^n}{n!}}{\frac{\lambda_0^{k_0} y^{k_0-1}}{(k_0-1)!} \sum_{n=0}^{k_2-1} \frac{(\lambda_2 y)^n}{n!} + \frac{\lambda_2^{k_2} y^{k_2-1}}{(k_2-1)!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 y)^n}{n!}},$$

$$\mathrm{E}(T_1 \mid x, y, \Lambda) = x, \text{ and}$$

$$\mathrm{E}(T_2 \mid x, y, \Lambda) = \frac{y f_{T_2}(y) \overline{F}_{T_0}(y) + f_{T_0}(y) \frac{k_2}{\lambda_2} e^{-\lambda_2 y} \sum_{n=0}^{k_2} \frac{(\lambda_2 y)^n}{n!}}{f_Y(y)}$$

$$= \frac{\frac{\lambda_2^{k_2} y^{k_2}}{(k_2-1)!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 y)^n}{n!} + \frac{\lambda_0^{k_0} y^{k_0-1}}{(k_0-1)!} \frac{k_2}{\lambda_2} \sum_{n=0}^{k_2} \frac{(\lambda_2 y)^n}{n!}}{\frac{\lambda_2^{k_2} y^{k_2-1}}{(k_2-1)!} \sum_{n=0}^{k_0-1} \frac{(\lambda_0 y)^n}{n!} + \frac{\lambda_0^{k_0} y^{k_0-1}}{(k_0-1)!} \sum_{n=0}^{k_2-1} \frac{(\lambda_2 y)^n}{n!}}.$$

Case 3. Assume $x = y > 0$. Then $\mathrm{E}(T_0 \mid x, y, \Lambda) = x$ and $T_1,\ T_2 \geq x$. The conditional density of T_1 is

$$f_{T_1|X,Y,\Lambda}(t \mid x, y, \Lambda) = \frac{f_{T_1}(t) f_{T_0}(x) \overline{F}_{T_2}(x)}{f_{X,Y}(x,x)} \quad \text{for } t \geq x,$$

and the conditional expectation is

$$\begin{aligned}
\mathrm{E}(T_1 \mid x, y, \Lambda) &= \int\limits_x^{\infty} t f_{T_1|X,Y,\Lambda}(t \mid x, y, \Lambda) dt \\
&= \frac{f_{T_0}(x)\overline{F}_{T_2}(x)}{f_{T_0}(x)\overline{F}_{T_1}(x)\overline{F}_{T_2}(x)} \frac{k_1}{\lambda_1} e^{-\lambda_1 x} \sum_{n=0}^{k_1} \frac{(\lambda_1 x)^n}{n!} \\
&= \frac{k_1}{\lambda_1}\left(1 + \frac{\lambda_1^{k_1} x^{k_1}}{k_1! \sum\limits_{n=0}^{k_1-1} \frac{\lambda_1^n x^n}{n!}}\right).
\end{aligned}$$

Similarly, the conditional expectation of T_2 is expressed as

$$\mathrm{E}(T_2 \mid x, y, \Lambda) = \frac{k_2}{\lambda_2}\left(1 + \frac{\lambda_2^{k_2} x^{k_2}}{k_2! \sum\limits_{n=0}^{k_2-1} \frac{\lambda_2^n x^n}{n!}}\right).$$

3.7.2. Simulation results

In this section, we present the performance of the aforementioned EM algorithm for different simulated data sets from the bivariate Erlang distributions with $k_0 = k_1 = k_2 = 2$ assumed known. All the simulations were carried out with the software MATLAB.

We generated samples of sizes $n = 25, 100, 500$ for the following choices of the parameters: $\lambda_0 = \lambda_1 = \lambda_2 = 1$; $\lambda_0 = 1, \lambda_1 = 2, \lambda_2 = 1$; $\lambda_0 = 2, \lambda_1 = 1, \lambda_2 = 5$, and $\lambda_0 = 0.5, \lambda_1 = 1, \lambda_2 = 2$. In each case, we used the corresponding true λ's as initial values for the EM algorithm and used as stopping criterion $\mid \lambda_i^{(k)} - \lambda_i^{(k+1)} \mid < 10^{-6}$, where $\lambda_i^{(k)}$, for $i = 0, 1, 2$, is the estimate after the k-th iteration. Table 1 illustrates the average estimates based on 100 replications, the standard error in brackets and the mean number of iterations for each case mentioned above.

We remark that as the sample size increases the parameters means are improved and the number of iterations required decreases. For each case, we also

Table 3.1. Parameters estimates (s.e.) and average number of iterations (AI) with $k_0 = k_1 = k_2 = 2$.

	λ_0	λ_1	λ_2	AI
Initial values	1	1	1	
n=25	1.0421 (0.1065)	1.0517 (0.1184)	1.0266 (0.1019)	23.84
n=100	1.0127 (0.0124)	1.0213 (0.0169)	1.0079 (0.0144)	20.75
n=500	1.0031 (0.0038)	1.0052 (0.0043)	1.0022 (0.0047)	17.68
Initial values	1	2	1	
n=25	1.0324 (0.0955)	2.0491 (0.1014)	1.0169 (0.1126)	28.33
n=100	1.0151 (0.0121)	2.0197 (0.0228)	0.9881 (0.0143)	26.1
n=500	1.0032 (0.0075)	2.0081 (0.0066)	0.9974 (0.0063)	23.85
Initial values	2	1	5	
n=25	1.9466 (0.1081)	1.0425 (0.0932)	4.9674 (0.1047)	93.25
n=100	1.9668 (0.0307)	1.0207 (0.0249)	4.9826 (0.0352)	81.11
n=500	1.9907 (0.0112)	1.0024 (0.0115)	4.9981 (0.0106)	69.25
Initial values	0.5	1	2	
n=25	0.4833 (0.1219)	1.0294 (0.1207)	2.0317 (0.1198)	60.8
n=100	0.4911 (0.0536)	1.0172 (0.0532)	2.0205 (0.0441)	52.44
n=500	0.4953 (0.0033)	1.0067 (0.0029)	2.0083 (0.0037)	42.63

tried some other initial values but the average estimates did not change. We noted that the number of iterations did increase by an average of 15 percent.

We have also generated data sets from the bivariate Erlang distributions assuming other values for k_0, k_1, and k_2, such as $(k_0, k_1, k_2) = (2, 2, 3)$ and $(k_0, k_1, k_2) = (3, 4, 5)$, and the performance of the EM algorithm is similar to that for the above case.

3.8. Conclusions

In this chapter, we introduced a class of bivariate Erlang (BVEr) distributions of Marshall-Olkin type. The bivariate exponential distribution introduced by Marshall-Olkin (1967a) is a particular case of this bivariate Erlang distribution. The BVEr distribution is not absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}_+^2$, because it has a positive probability on the $x = y$ axis. It has been shown that the marginals and their minimum follow finite mixtures of Erlang distributions with common rate parameter, while their maximum is a weighted average of Erlang distributions. The joint probability density function, conditional probability density function, conditional expectation and Laplace transform have been derived. We extended this distribution to the bivariate model considering mixtures of Erlang distributions and we obtained a mixture of BVEr distributions. For the BVEr distribution, we obtained the maximum likelihood estimates via an EM algorithm assuming that the shape parameters are known.

The distributions we introduced in this thesis have potential applications in various fields such as the actuarial science, finance, reliability theory and medical studies.

Chapter 4

RUIN PROBABILITIES IN A MULTIVARIATE POISSON MODEL

4.1. Introduction

In an insurance company, for a given portfolio of insurance policies of different types such as health, automobile, or house insurance, it is advantageous to have an accurate forecast of the expected liability of these policies. In practice, there are situations in which the assumption of independent policies is not verified. For example, in the case of a catastrophe such as an earthquake, the damages covered by homeowners and private passenger automobile insurance cannot be considered independent. Therefore, it is desirable to develop models which assume that different policies are dependent in order to increase the accuracy of the estimation of the costs associated to different policies.

These reasons motivate us to consider investigating multivariate risk processes which may be useful in studying ruin problems for insurance companies handling dependent classes of business. A review of the results related to different ruin concepts in a multivariate setting, which were introduced in Definition 2.4.1, was presented in Subsection 2.4.2.

We assume that an insurance company has m different classes of insurance business allowing for dependence between claim sizes and dependence among the numbers of claims across classes.

The aim of our study is to reformulate this m-dimensional risk model in terms of a piecewise deterministic Markov (PDM) process and derive suitable martingales via Proposition 2.1.5, which gives a connection between Markov processes and martingales. These martingales are further used in establishing computable bounds for the multivariate ruin probabilities whose expressions are intractable.

In the same Markovian framework, we also obtain martingales for the situation where a Brownian perturbation is added to each class of business with a joint correlation matrix.

The PDM processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models.

As mentioned in Chapter 2, Dassios and Embrechts (1989) showed in general how to use this theory to solve insurance risk problems dealing with univariate models and since then, the martingale technique via PDM processes has become a systematic approach in dealing with continuous-time risk models. See, for example, Davis (1993), Rolski et al. (1999), Asmussen (2000), Dassios and Jang (2003), Jang (2004, 2007), Liu et al. (2005), Lu et al. (2007), Schmidli (2010).

Inspired by Dassios and Embrechts (1989), we complement their work by employing tools from the piecewise deterministic Markov processes theory for the multivariate risk models in order to obtain Lundberg-type upper bounds for ruin probabilities.

Our contributions in this chapter are:

• We use the Poisson model with common shocks described in (2.4.11) as a method to model the dependence between the number of claims. By this approach, we extend the model proposed by Asmussen and Albrecher (2010) and illustrated by Proposition 2.4.10. The reason for considering this scenario is to show how the ruin probability is affected by just having independent Poisson processes for claims number processes to then gradually adding common shocks that have impact on couples of classes and finally, on all m classes of insurance business. This model would illustrate more realistic situations, as we will see in Section 4.2.

• We apply results from the theory of PDM processes in order to derive exponential martingales needed for establishing upper bounds for the probability that ruin occurs in all classes simultaneously, denoted by $\psi_{sim}(u_1, ..., u_m)$ and introduced in Definition 2.4.1.

• We derive two new results, namely, an expression for the probability that ruin occurs in at least one class of business, denoted by $\psi_{or}(u_1, ..., u_m)$ and introduced in Definition 2.4.1, and an asymptotic upper bound for the finite-time ruin probability $\psi_{sim}(u_1, ..., u_m, t)$ in the case of dependent heavy-tailed claims.

• By adding an m-dimensional Brownian motion to the aforementioned multivariate risk process, we extend the model proposed by Li, Liu and Tang (2007) and illustrated by Proposition 2.4.8. By assuming that, for $m \geq 3$, the correlation coefficients between the components of the diffusion process are non-negative, except for at most one element, we derive an upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$ with the aid of an exponential martingale obtained via Proposition 2.1.5.

This chapter is structured as follows.

In Section 4.2, we give a description of the m-dimensional risk model associated to m classes of business, where the claims arrivals are assumed to be dependent Poisson processes with common shocks, as in (2.4.11). In Sections 4.3 and 4.4 respectively, we derive an exponential martingale related to this continuous-time m-dimensional risk process with the aid of the tools from PDM processes theory and based on this, the corresponding Lundberg upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$ is obtained.

In Section 4.5, we obtain an expression for the ruin probability $\psi_{or}(u_1, ..., u_m)$, which in the particular case $m = 1$ gives formula (2.3.6) of Proposition 2.3.2.

In Section 4.6, we investigate the asymptotic behavior of the finite-time ruin probability of type $\psi_{sim}(u_1, ..., u_m, t)$ in the case of dependent heavy-tailed claims and an asymptotic upper estimate is derived as the initial surplus for each class of business increases. To complete this section, we obtain an asymptotic estimate of this ruin probability in the case of independent heavy-tailed claims, as an extension of the result illustrated by Proposition 2.3.15 for the classical risk model.

In Section 4.7, we consider studying the m-dimensional risk model, introduced in Section 4.2, perturbed by a diffusion which is characterized by an m-dimensional correlated Brownian motion. Our approach to derive an exponential martingale is based on using Proposition 2.1.5, and in this framework, an upper bound of the ruin probability $\psi_{sim}(u_1, ..., u_m)$ is obtained.

In Section 4.8, we present numerical results for the upper bounds obtained in Sections 4.4 and 4.7 assuming the trivariate case. We choose this particular situation with three classes of insurance business since it contains all types of common shocks involved in the model proposed for the frequency of claims given in Section 4.2. For modeling the dependence between claim sizes across classes we use the notion of copula, which allows the construction of multivariate distributions with arbitrary marginal laws. Due to its popularity in the literature and simplicity, a member of Farlie-Gumbel-Morgenstern family copula with exponential marginals is used in the numerical illustrations. In this section, a discussion of the numerical results obtained is provided.

Section 4.9 concludes the chapter.

4.2. Multivariate risk model formulation

Consider an insurance company with $m \geq 1$ possibly dependent classes of business. The surplus process $\{U_i(t), t \geq 0\}$ of the i-th class of business is described by

$$U_i(t) = u_i + c_i\, t - \sum_{k=1}^{N_i(t)} X_{ik}, \quad t \geq 0, \quad i = 1, 2, .., m, \tag{4.2.1}$$

where the initial surplus and premium rate are denoted by u_i and c_i, respectively, provided that $u_i \geq 0$ and $c_i > 0$. In each class, the claim arrivals and claim sizes are modeled respectively by the counting process $N_i(t)$ and the positive random variables X_{ik}'s with $k \geq 1$ assuming that $\{N_i(t), t \geq 0\}$ and $\{X_{ik}\}_{k \geq 1}$ are independent.

In what follows, we give a description of the dependence structure incorporated in the risk model (4.2.1).

Suppose that $\{(X_{1k}, ..., X_{mk})\}_{k\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) m-dimensional random vectors; for simplicity, let $(X_1, ..., X_m)$ be an arbitrary random vector from $\{(X_{1k}, ..., X_{mk})\}_{k\geq 1}$.

We assume that $F(x_1, ..., x_m)$ is a continuous joint distribution function of the vector $(X_1, ..., X_m)$ and for $i, j = 1, ..., m$, $i \neq j$, $F_{i,j}(x_i, x_j)$ stands for the joint distribution function of (X_i, X_j), while the random variables X_i, $i = 1, ..., m$, have the distribution functions $F_i(x_i)$ satisfying $F_i(0) = 0$ and with finite mean $\mu_i = E[X_i]$. In the case where for each $k = 1, 2, ...$, the random variables X_{1k}, X_{2k},...,X_{mk} are all mutually independent, then $F(x_1, ..., x_m) = \prod_{i=1}^{m} F_i(x_i)$ and $F_{i,j}(x_i, x_j) = F_i(x_i)F_j(x_j)$ for $i, j = 1, ..., m$, $i \neq j$.

We further assume that the claims number processes $\{N_i(t), t \geq 0\}$, for $i = 1, ..., m$, follow a Poisson model with common shocks as was defined in (2.4.11). Recall that this model means that in addition to the individual shocks, a common shock affects the m classes of business and that another common shock has an impact on each couple of classes. Mathematically, $\{N_i(t), t \geq 0\}$, $i = 1, ..., m$, are defined as follows:

$$\begin{aligned}
N_1(t) = &\quad N_{11}(t) + N_{12}(t) + ... + N_{1m}(t) + N_{1...m}(t), \\
\cdot &\qquad\qquad\qquad \cdot \\
\cdot &\qquad\qquad\qquad \cdot \\
\cdot &\qquad\qquad\qquad \cdot \\
N_m(t) = &\quad N_{mm}(t) + N_{1m}(t) + ... + N_{m-1\ m}(t) + N_{1...m}(t),
\end{aligned}$$

where $\{N_{ij}(t), t \geq 0\}$, $(1 \leq i, j \leq m)$ and $\{N_{1...m}(t), t \geq 0\}$ are all mutually independent Poisson processes with parameters λ_{ij} and $\lambda_{1...m}$, respectively, and $\lambda_{ij} = \lambda_{ji} \geq 0$ for $i \neq j$.

Therefore, the common shock which affects all classes of business is arriving according to the Poisson process $N_{1...m}(t)$, while the occurrence of the common shocks that affect couples of classes of business are governed by the Poisson processes denoted by $\{N_{ij}(t), t \geq 0\}$, $(1 \leq i < j \leq m)$ and the individual shocks arrive according to the Poisson processes $\{N_{ii}(t), t \geq 0\}$. In fact, this is a particular case of the general multiple common shocks model which assumes that each shock yields claims in a different subset of the m lines.

Typical lines of business in an insurance company are automobile insurance, house insurance, health, disability or liability. In some cases, claims for different classes of business emerge from a common event: for example, a car accident may cause a claim for automobile insurance, liability and disability insurance.

Also, common shocks may affect couples of classes of insurance. For instance, a strong wind-storm or an earthquake will be likely to make claims on both automobile and house policies, or on both automobile and health policies, or on both house and health policies. This should correspond to simultaneous jumps across classes in the multivariate process. One of the models that incorporate this kind of dependence is the Poisson model with common shocks (2.4.11), which has been studied by many authors such as Ambagaspitiya (1998, 1999, 2003), Cossette and Marceau (2000), Wang (1998), and Yuen et al. (2002, 2006).

Using property (5) in Proposition 2.2.4, we have that the processes $\{N_i(t), t \geq 0\}$, for $i = 1, ..., m$, are Poisson processes with respective parameters

$$\lambda_i = \lambda_{i1} + \lambda_{i2} + ... + \lambda_{im} + \lambda_{1...m}.$$

In this model, the dependence is due to the common arrival processes $N_{ij}(t)$, $(1 \leq i < j \leq m)$, $N_{1...m}(t)$ together with the dependence between claim sizes across classes of insurance business.

According to Definitions 2.2.2 and 2.2.4 from Section 2.2, the time of ruin for the i-th class $(i = 1, ..., m)$ is defined as

$$\tau_i = \inf\{t \geq 0 : U_i(t) < 0\},$$

and the corresponding ruin probability as

$$\psi_i(u_i) = P(\tau_i < \infty \mid U_i(0) = u_i).$$

If for each $i = 1, ..., m$, the surplus $U_i(t) \geq 0$ for all $t \geq 0$ (no ruin occurs), we indicate this by writing $\tau_i = \infty$.

In order that ruin does not occur almost surely, the net profit condition (2.3.1) is satisfied for each class of business, that is, the premium rate will exceed the expected aggregate claims per unit time:

$$c_i > \left(\sum_{j=1}^{m} \lambda_{ij} + \lambda_{1...m} \right) \mu_i, \quad i = 1, ..., m. \tag{4.2.2}$$

Thus, the random variable τ_i may be defective; that is, it may happen that $P(\tau_i < \infty) < 1$.

In Section 2.4, different ruin concepts in a multivariate setting were introduced via Definition 2.4.1. As it was exemplified in Subsection 2.4.2, the problem involving the ruin probability of type ψ_{sum} can be reduced to a one-dimensional ruin problem. In the general case of having m classes of business, given by (4.2.1), we present the process corresponding to the ruin probability ψ_{sum} as follows.

Note that the result from Proposition 2.4.1 can be extended to an m-variate ($m > 2$) claim number process whose component is a linear combination of independent Poisson processes. Henceforth, in the multivariate framework defined by (4.2.1), the ruin probability ψ_{sum} is associated to the risk process

$$U(t) = U_1(t) + ... + U_m(t),$$

which is distributed as the process

$$U'(t) = u + ct - \sum_{k=1}^{M(t)} X'_k, \tag{4.2.3}$$

where $u = u_1 + ... + u_m$, $c = c_1 + ... + c_m$, $M(t)$ is a Poisson process with parameter $\lambda = \sum_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1...m}$ independent of $\{X'_k\}_{k \geq 1}$. Moreover, $\{X'_k\}_{k \geq 1}$ is a sequence of independent and identically distributed random variables with the common distribution function described as

$$F_{X'}(x) = \sum_{i=1}^{m} \frac{\lambda_{ii}}{\lambda} F_{X_i}(x) + \sum_{1 \leq i < j \leq m} \frac{\lambda_{ij}}{\lambda} F_{X_i + X_j}(x) + \frac{\lambda_{1...m}}{\lambda} F_{X_1 + ... + X_m}(x),$$

where $F_{X_i}(x) = F_i(x) = P(X_i \leq x)$, $F_{X_i+X_j}(x) = P(X_i + X_j \leq x)$ and $F_{X_1+...+X_m}(x) = P(X_1 + ... + X_m \leq x)$ represent the distribution functions of X_i, $X_i + X_j$ and $X_1 + ... + X_m$, respectively. Thus, the risk model given by (4.2.3) represents the Cramér-Lundberg model presented in Section 2.3.1, and classical ruin probability results can be applied.

Since we are interested in a multivariate model, we turn our attention to the other types of ruin probabilities from Definition 2.4.1.

The system performance measures that we examine are the ruin probabilities of type ψ_{sim}, which denotes the probability that ruin occurs in all classes simultaneously or at the same instant in time, and ψ_{or}, which is the probability that ruin occurs in at least one class of business, both probabilities being described by Definition 2.4.1. By comparing τ_{or} and τ_{sim}, we note that τ_{sim} represents a more critical time for the insurance company because at time τ_{or}, probably only one of the classes of business gets ruined. As we will see throughout this chapter, the techniques applied for obtaining bounds for these types of ruin probabilities can not be applied for the ruin probability ψ_{and}.

We start by reformulating the risk model (4.2.1) in terms of a piecewise deterministic Markov process and then derive its infinitesimal generator. This is followed by the derivation of an exponential martingale needed in our ruin problem. These steps are covered in the next section.

4.3. Infinitesimal generator and martingales

In this section, we derive exponential martingales related to the multivariate risk process given by relation (4.2.1) using tools from the theory of piecewise deterministic Markov (PDM) processes. The latter, presented in Section 2.1, are Markov processes consisting of a mixture of a deterministic motion and random jumps. These martingales will be used in obtaining an upper bound of Lundberg type for the ruin probability $\psi_{sim}(u_1, ..., u_m)$ and an expression for the ruin probability $\psi_{or}(u_1, ..., u_m)$. These results are presented in Sections 4.4 and 4.5, respectively.

For each $i = 1, ..., m$, let $M_{X_i}(r_i)$ be the moment generating function (m.g.f.) of X_i. Similarly to the classical risk model, let us consider the case of light-tailed marginal claim size distributions. Therefore, assume that there exists $0 < r_i^0 \leq \infty$ such that $M_{X_i}(r_i) < \infty$ for all $r_i < r_i^0$ and $\lim_{r_i \uparrow r_i^0} M_{X_i}(r_i) = \infty$.

For $i, j = 1, ..., m$, $i \neq j$, the joint moment generating function of (X_i, X_j) is defined by $M_{X_i,X_j}(r_i, r_j) = E[e^{(r_i X_i + r_j X_j)}]$ and that of $(X_1, ..., X_m)$ is defined by $M_{X_1,...,X_m}(r_1, ..., r_m) = E[e^{(r_1 X_1 + ... + r_m X_m)}]$. Clearly, $M_{X_i,X_j}(0, 0) = 1$ and $M_{X_1,...,X_m}(0, ..., 0) = 1$.

Let us define the following sets:

$$M_{ij} = \{(r_i, r_j) \in [0, r_i^0) \times [0, r_j^0) \mid M_{X_i,X_j}(r_i, r_j) < \infty\} - \{(0, 0)\},$$

where $i, j = 1, ..., m$, $i \neq j$, and

$$M = \{(r_1, .., r_m) \in [0, r_1^0) \times ... \times [0, r_m^0) \mid M_{X_1,..,X_m}(r_1, .., r_m) < \infty\} - \{(0, .., 0)\}.$$

Then, we have the following result.

Lemma 4.3.1. *The sets M_{ij} and M are non-empty.*

Proof. Since $M_{X_i,X_j}(r_i, r_j) = M_{X_1,..,X_m}(0, ..., r_i, ..., r_j, ..., 0)$, it is sufficient to prove that the set M is non-empty. In order to prove that the set M is non-empty, we use the generalized Hölder's inequality given below.

For any positive random variables X_i, $i = 1, ..., n$,

$$E[X_1 ... X_n] \leq E[X_1^{\alpha_1}]^{1/\alpha_1} \cdot ... \cdot E[X_n^{\alpha_n}]^{1/\alpha_n}, \tag{4.3.1}$$

where numbers $\alpha_1 > 0$,...,$\alpha_n > 0$ satisfy $\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_n} = 1$. For the proof of this inequality we refer to Halmos (1978).

If α_1,...,α_m are strictly positive numbers such that $\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_m} = 1$, then choosing $0 \leq r_i < r_i^0/\alpha_i$, $i = 1, ..., m$, with $(r_1, ..., r_m) \neq (0, ..., 0)$ implies

$$M_{X_1,...,X_m}(r_1, ..., r_m) \leq \prod_{i=1}^{m} \left(E[e^{\alpha_i r_i X_i}]\right)^{1/\alpha_i} = \prod_{i=1}^{m} [M_{X_i}(r_i\alpha_i)]^{1/\alpha_i} < \infty,$$

where inequality (4.3.1) was used for establishing the first inequality and the property $M_{X_i}(r_i\alpha_i) < \infty$ for $r_i\alpha_i < r_i^0$ was used for the second inequality. □

Assume that the vector process $\{(U_1(t), ..., U_m(t)), t \geq 0\}$ is defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\mathcal{F}_t = \mathcal{F}_t^{U_1} \vee ... \vee \mathcal{F}_t^{U_m}$ and $\mathcal{F}_t^{U_i}$ is the natural filtration of the process $\{U_i(t), t \geq 0\}$ described as $\mathcal{F}_t^{U_i} = \sigma(U_i(s), 0 \leq s \leq t)$, for $i = 1, 2, ..., m$.

In Subsection 2.3.4, it was established that for $i = 1, ..., m$, the surplus processes $\{U_i(t), t \geq 0\}$ are PDM processes. Therefore, $\{(U_1(t), ..., U_m(t)), t \geq 0\}$ is a piecewise deterministic Markov vector process which, in between jumps (claims), evolves deterministically as

$$(U_1(t), ..., U_m(t)) = (z_1 + c_1 t, ..., z_m + c_m t),$$

for some values z_1,...,z_m. Let us define

$$N(t) = \sum_{1\leq i\leq j\leq m} N_{ij}(t) + N_{1...m}(t). \tag{4.3.2}$$

According to property (5) of Proposition 2.2.4, $N(t)$ is an homogeneous Poisson process with rate $\lambda = \sum_{1\leq i\leq j\leq m} \lambda_{ij} + \lambda_{1...m}$. Note that the jump times of $(U_1(t), ..., U_m(t))$ are precisely the jump times of $N(t)$. We proceed to derive the infinitesimal generator of $\{(U_1(t), ..., U_m(t), t), t \geq 0\}$, which is given by the following proposition.

Proposition 4.3.1. *The infinitesimal generator of the homogeneous Markov vector process $(U_1(t), ..., U_m(t), t)$ acting on a function $f(z_1, ..., z_m, t)$ belonging to its domain is described as*

$$\mathcal{A}f(z_1, .., z_m, t) = \sum_{i=1}^{m} c_i \frac{\partial f(z_1, .., z_m, t)}{\partial z_i} + \frac{\partial f(z_1, .., z_m, t)}{\partial t}$$
$$+ \sum_{i=1}^{m} \lambda_{ii} \left[\int_0^\infty f(z_1, ..., z_i - x_i, ..., z_m, t) dF_i(x_i) - f(z_1, ..., z_m, t) \right]$$
$$+ \sum_{1\leq i<j\leq m} \lambda_{ij} \left[\int_{[0,\infty)^2} f(z_1, ., z_i - x_i, ., z_j - x_j, .., z_m, t) dF_{i,j}(x_i, x_j) - f(z_1, .., z_m, t) \right]$$
$$+ \lambda_{1...m} \left[\int_{[0,\infty)^m} f(z_1 - x_1, ..., z_m - x_m, t) dF(x_1, ..., x_m) - f(z_1, ..., z_m, t) \right], \tag{4.3.3}$$

where $f : \mathbb{R}^m \times (0, \infty) \to (0, \infty)$ is differentiable with respect to z_1,...,z_m, t for all z_1,...,z_m, t.

Proof. By Definition 2.1.6, we have that $\mathcal{A}f(z_1, .., z_m, t)$ is equal to

$$\lim_{h\downarrow 0} \frac{E[f(U_1(t+h), .., U_m(t+h), t+h) \mid U_i(t) = z_i, i = 1, ..., m] - f(z_1, .., z_m, t)}{h}, \tag{4.3.4}$$

where the domain of $\mathcal{A}$ is the set of all measurable functions f for which this limit exists. Since

$$E[f(U_1(t+h), .., U_m(t+h), t+h) \mid U_i(t) = z_i, i = 1, ..., m]$$
$$= E[f(z_1 + c_1 h - \sum_{k=N_1(t)+1}^{N_1(t+h)} X_{ik}, .., z_m + c_m h - \sum_{k=N_m(t)+1}^{N_m(t+h)} X_{ik}, t+h) \mid U_i(t) = z_i], \tag{4.3.5}$$

for the small time interval $(t, t+h]$ we consider the following possible cases:

1. no claim occurs in $(t, t+h]$: $N(t+h) - N(t) = 0$,

2. only one claim occurs in $(t, t+h]$: $N(t+h) - N(t) = 1$, and

3. more than one claim occurs in $(t, t+h]$: $N(t+h) - N(t) \geq 2$,

where $N(t)$ is defined by (4.3.2).

In view of the property (2) given by Proposition 2.2.4, and of the fact that a claim can occur due to any of the $m + m(m-1)/2 + 1 = (m^2 + m + 2)/2$ independent Poisson processes, the probability of the event in case 1 is

$$e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h},$$

in case 2 is

$$\sum_{1 \leq i \leq j \leq m} \lambda_{ij} h e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h} + \lambda_{1\ldots m} h e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h},$$

and in case 3 is $o(h)$. Therefore, by using the property that the compound Poison processes $\sum\limits_{k=1}^{N_i(t)} X_{ik}$, $i = 1, \ldots, m$, have independent and stationary increments in relation (4.3.5), and by the total law of probability, the limit (4.3.4) is equal to

$$\lim_{h \downarrow 0} \frac{e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h} f(z_1 + c_1 h, \ldots, z_m + c_m h, t+h) - f(z_1, \ldots, z_m, t)}{h}$$

$$+ \lim_{h \downarrow 0} \sum_{i=1}^{m} \frac{\lambda_{ii} h e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h}}{h}$$

$$\times \int_0^\infty f(z_1 + c_1 h, \ldots, z_i + c_i h - x_i, \ldots, z_m + c_m h, t+h) dF_i(x_i)$$

$$+ \lim_{h \downarrow 0} \sum_{1 \leq i < j \leq m} \frac{\lambda_{ij} h e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h}}{h}$$

$$\times \int_{[0,\infty)^2} f(z_1 + c_1 h, ., z_i + c_i h - x_i, .., z_j + c_j h - x_j, .., z_m + c_m h, t+h) dF_{i,j}(x_i, x_j)$$

$$+ \lim_{h \downarrow 0} \frac{\lambda_{1\ldots m} h e^{-\left[\sum\limits_{1 \leq i \leq j \leq m} \lambda_{ij} + \lambda_{1\ldots m}\right] h}}{h}$$

$$\times \int\limits_{[0,\infty)^m} f(z_1+c_1h-x_1,...,z_m+c_mh-x_m,t+h)dF(x_1,...,x_m)+\lim_{h\downarrow 0}\frac{o(h)}{h}. \quad (4.3.6)$$

The first term of (4.3.6) corresponds to case 1, the next three terms correspond to case 2 and the last term corresponds to case 3. For the first limit of (4.3.6), using a Taylor series' expansion for the function $f(z_1+h,..,z_m+h,t+h)$ at $(z_1,..,z_m,t)$ leads to the following result

$$\lim_{h\downarrow 0}\frac{e^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1...m}\right]h}-1}{h}\, f(z_1,...,z_m,t)$$

$$+\sum_{i=1}^{m}c_i\frac{\partial f(z_1,...,z_m,t)}{\partial z_i}+\frac{\partial f(z_1,...,z_m,t)}{\partial t}$$

$$=-\left[\sum_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1...m}\right]\, f(z_1,...,z_m,t)+\sum_{i=1}^{m}c_i\frac{\partial f(z_1,...,z_m,t)}{\partial z_i}+\frac{\partial f(z_1,...,z_m,t)}{\partial t}, \quad (4.3.7)$$

since

$$\lim_{h\downarrow 0}\frac{e^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1...m}\right]h}-1}{h}=\frac{d}{dx}e^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1...m}\right]x}\Big|_{x=0}=-\sum_{1\le i\le j\le m}\lambda_{ij}-\lambda_{1...m}. \quad (4.3.8)$$

The second limit of (4.3.6) is equal to

$$\sum_{i=1}^{m}\lambda_{ii}\left[\int\limits_{0}^{\infty} f(z_1,...,z_i-x_i,...,z_m,t)dF_i(x_i)\right], \quad (4.3.9)$$

the third limit of (4.3.6) is equal to

$$\sum_{1\le i<j\le m}\lambda_{ij}\left[\int\limits_{[0,\infty)^2} f(z_1,...,z_i-x_i,...,z_j-x_j,...,z_m,t)dF_{i,j}(x_i,x_j)\right], \quad (4.3.10)$$

and the fourth limit of (4.3.6) is equal to

$$+\lambda_{1...m}\left[\int\limits_{[0,\infty)^m} f(z_1-x_1,...,z_m-x_m,t)dF(x_1,...,x_m)\right]. \quad (4.3.11)$$

Substituting (4.3.7), (4.3.9), (4.3.10) and (4.3.11) into (4.3.6) yields the relation (4.3.3).

For $f(z_1, ..., z_m, t)$ to belong to the domain of the generator $\mathcal{A}$, that is, the limit (4.3.4) exists, it is sufficient that $f(z_1, ..., z_m, t)$ be differentiable with respect to z_1,...,z_m, t for all z_1,...,z_m, t, and that

$$\left| \int_0^\infty f(z_1, ..., z_i - x_i, ..., z_m, t) dF_i(x_i) - f(z_1, ..., z_m, t) \right| < \infty, \; i = 1, .., m, \quad (4.3.12)$$

$$\left| \int_{[0,\infty)^2} f(z_1, .., z_i - x_i, .., z_j - x_j, .., z_m, t) dF_{i,j}(x_i, x_j) - f(z_1, .., z_m, t) \right| < \infty, \quad (4.3.13)$$

with $1 \le i < j \le m$, and

$$\left| \int_{[0,\infty)^m} f(z_1 - x_1, ..., z_m - x_m, t) dF(x_1, ..., x_m) - f(z_1, ..., z_m, t) \right| < \infty. \quad (4.3.14)$$

The proof is completed. $\square$

The following theorem gives the construction of a martingale using the infinitesimal generator established in Proposition 4.3.1.

Theorem 4.3.1. *If* $(r_1, ..., r_m) \in M$*, then the process*

$$Z(t) = e^{-tg(r_1,...,r_m)} e^{-r_1 U_1(t) - ... - r_m U_m(t)}, \; t \ge 0,$$

is a martingale with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, P)$*, where* $g(r_1, ..., r_m)$ *is defined as follows*

$$g(r_1, ..., r_m) = -\sum_{i=1}^m c_i r_i + \sum_{i=1}^m \lambda_{ii} \left[M_{X_i}(r_i) - 1 \right]$$

$$+ \sum_{1 \le i < j \le m} \lambda_{ij} \left[M_{X_i, X_j}(r_i, r_j) - 1 \right] + \lambda_{1...m} [M_{X_1, ..., X_m}(r_1, ..., r_m) - 1]. \quad (4.3.15)$$

Proof. According to Proposition 2.1.5, we have that for a function f belonging to the domain of the infinitesimal generator described by relation (4.3.3) such that $\mathcal{A}f = 0$, the process $\{f(U_1(t), ..., U_m(t), t), t \ge 0\}$ is a martingale. The equation $\mathcal{A}f = 0$ is equivalent to

$$\sum_{i=1}^m c_i \frac{\partial f(z_1, ..., z_m, t)}{\partial z_i} + \frac{\partial f(z_1, ..., z_m, t)}{\partial t}$$

$$+\sum_{i=1}^{m}\lambda_{ii}\left[\int_{0}^{\infty} f(z_1,...,z_i-x_i,...,z_m,t)dF_i(x_i)-f(z_1,...,z_m,t)\right]$$

$$+\sum_{1\le i<j\le m}\lambda_{ij}\left[\int_{[0,\infty)^2} f(z_1,.,z_i-x_i,..,z_j-x_j,.,z_m,t)dF_{i,j}(x_i,x_j)-f(z_1,...,z_m,t)\right]$$

$$+\lambda_{1...m}\left[\int_{[0,\infty)^m} f(z_1-x_1,...,z_m-x_m,t)dF(x_1,...,x_m)-f(z_1,...,z_m,t)\right]=0. \tag{4.3.16}$$

In fact, we need only a special solution. Intuitively, an exponential form of the solution is suitable for this equation and convenient for conditions (4.3.12), (4.3.13), and (4.3.14) to be satisfied, knowing that the sets M_{ij} and M are non-empty. Motivated by Dassios and Embrechts (1989), we try a solution of the form $f(z_1,...,z_m,t)=\alpha(t)e^{-r_1z_1-...-r_mz_m}$, where we can assume that $\alpha(0)=1$. Then the equation (4.3.16) yields

$$\alpha'(t)+\alpha(t)g(r_1,...,r_m)=0,$$

where $g(r_1,...,r_m)$ is expressed as (4.3.15), and the solution is

$$\alpha(t)=e^{-tg(r_1,...,r_m)}.$$

Therefore, the process $Z(t)=e^{-tg(r_1,...,r_m)}e^{-r_1U_1(t)-...-r_mU_m(t)}$ is a martingale. □

Note that in the case of only one class of business, relation (4.3.15) becomes $g(r_1)=-c_1r_1+\lambda(M_{X_1}(r_1)-1)$. The equation $g(r_1)=0$ represents the Lundberg equation (2.3.2) which, as was stated in Subsection 2.3.1.1, admits a unique positive solution R, the so-called the adjustment coefficient.

Therefore, it is natural to analyze the existence of solutions for equation $g(r_1,...,r_m)=0$ for $m>1$. First, we present the following result.

Lemma 4.3.2. *If X is a continuous random variable such that $P(X>0)>0$, then $E[X^2]>0$.*

Proof. There exists a constant $a > 0$ such that $P(X \geq a) > 0$. Indeed, if $P(X \geq a) = 0$ for all $a > 0$, we can write

$$P(X > 0) = \lim_{a \downarrow 0} P(X \geq a) = 0,$$

which would contradict the hypothesis.
Now we obtain $E[X^2] \geq \int\limits_a^\infty x^2 dF_X(x) \geq a^2 P(X \geq a) > 0$. $\square$

As in Propositions 2.4.8 and 2.4.10, where the respective assumptions $\sup_{(s_1,s_2)\in G} f(s_1, s_2) > 0$ and $\sup_{(s_1,...,s_m)\in G} f(s_1, ..., s_m) > 0$ are helpful in proving that the sets of solutions for the corresponding equations: $f(s_1, s_2) = 0$ and $f(s_1, ..., s_m) = 0$ are non-empty, we impose a similar condition on $g(r_1, ..., r_m)$.

Recall from Subsection 2.3.1.1 that this assumption was not needed in the classical risk model since $\lim_{r\uparrow r_0} g(r) = \infty$ $(0 < r_0 \leq \infty)$ with $g(r)$ defined by (2.3.4).

Theorem 4.3.2. *If* $\sup_{(r_1,...,r_m)\in M} g(r_1, ..., r_m) > 0$, *then the equation* $g(r_1, ..., r_m) = 0$ *admits at least one solution in* M.

Proof. Note that $g(0, 0, ..., 0) = 0$ and we want to examine the sign of $g(r_1, ..., r_m)$ around the origin. For this, consider that k_2,...,k_m are given non-negative real numbers, and let us define

$$h(r_1) = g(r_1, k_2 r_1, ..., k_m r_1), \quad r_1 \in [0, r_1^0).$$

The first derivative of h is

$$\frac{dh(r_1)}{dr_1} = -c_1 - \sum_{i=2}^m c_i k_i + \lambda_{11} \frac{dM_{X_1}(r_1)}{dr_1} + \sum_{i=2}^m \lambda_{ii} k_i \frac{dM_{X_i}(r_i)}{dr_i} \Big|_{r_i = k_i r_1}$$

$$+ \sum_{j=2}^m \lambda_{1j} \left[\frac{\partial M_{X_1,X_j}(r_1, r_j)}{\partial r_1} + k_j \frac{\partial M_{X_1,X_j}(r_1, r_j)}{\partial r_j} \right] \Big|_{r_j = k_j r_1}$$

$$+ \sum_{2\leq i<j\leq m} \lambda_{ij} \left[k_i \frac{\partial M_{X_i,X_j}(r_i, r_j)}{\partial r_i} + k_j \frac{\partial M_{X_1,X_j}(r_i, r_j)}{\partial r_j} \right] \Big|_{r_i = k_i r_1, r_j = k_j r_1}$$

$$+\lambda_{1...m} \left[\frac{\partial M_{X_1,...,X_m}(r_1, ..., r_m)}{\partial r_1} + \sum_{i=2}^m k_i \frac{\partial M_{X_1,...,X_m}(r_1, ..., r_m)}{\partial r_i} \right] \Big|_{r_i = k_i r_1} .$$

Using the net profit condition (4.2.2) for each class, we obtain

$$\frac{dh(r_1)}{dr_1}|_{r_1=0} = -c_1 - \sum_{i=2}^{m} c_i k_i + \lambda_{11}\mu_1 + \sum_{i=2}^{m} \lambda_{ii} k_i \mu_i + \sum_{j=2}^{m} \lambda_{1j}\left[\mu_1 + k_j\mu_j\right]$$

$$+ \sum_{2\le i<j\le m} \lambda_{ij}\left[k_i\mu_i + k_j\mu_j\right] + \lambda_{1...m}\left[\mu_1 + \sum_{i=2}^{m} k_i\mu_i\right]$$

$$= \left[(\sum_{j=1}^{m} \lambda_{1j} + \lambda_{1...m})\mu_1 - c_1\right] + \sum_{i=2}^{m}\left[(\sum_{j=1}^{m} \lambda_{ij} + \lambda_{1...m})\mu_i - c_i\right] k_i < 0.$$

Therefore, $h(r_1)$ is a decreasing function at zero. The second derivative of $h(r_1)$ is given by

$$\frac{d^2h(r_1)}{dr_1^2} = \lambda_{11}\frac{d^2M_{X_1}(r_1)}{dr_1^2} + \sum_{i=2}^{m} \lambda_{ii} k_i^2 \frac{d^2M_{X_i}(r_i)}{dr_i^2} \ |_{r_i=k_ir_1}$$

$$+\sum_{j=2}^{m} \lambda_{1j}\{\frac{\partial^2 M_{X_1,X_j}(r_1,r_j)}{\partial r_1^2} + 2k_j\frac{\partial^2 M_{X_1,X_j}(r_1,r_j)}{\partial r_1\partial r_j} + k_j^2\frac{\partial^2 M_{X_1,X_j}(r_1,r_j)}{\partial r_j^2}\} \ |_{r_j=k_jr_1}$$

$$+\sum_{2\le i<j\le m} \lambda_{ij}\{2k_ik_j\frac{\partial^2 M_{X_i,X_j}(r_i,r_j)}{\partial r_i\partial r_j} + k_i^2\frac{\partial^2 M_{X_i,X_j}(r_i,r_j)}{\partial r_i^2} + k_j^2\frac{\partial^2 M_{X_i,X_j}(r_i,r_j)}{\partial r_j^2}\} \ |_{r_j=k_jr_1}$$

$$+\lambda_{1...m}\{\frac{\partial^2 M_{X_1,..,X_m}(r_1,..,r_m)}{\partial r_1^2} + 2\sum_{i=2}^{m} k_i\frac{\partial^2 M_{X_1,..,X_m}(r_1,..,r_m)}{\partial r_1\partial r_i}$$

$$+\sum_{i=2}^{m} k_i^2\frac{\partial^2 M_{X_1,..,X_m}(r_1,..,r_m)}{\partial r_i^2} + 2\sum_{2\le i<j\le m} k_ik_j\frac{\partial^2 M_{X_1,..,X_m}(r_1,..,r_m)}{\partial r_i\partial r_j}\} \ |_{r_i=k_ir_1},$$

and by Lemma 4.3.2, it follows that

$$\frac{d^2h(r_1)}{dr_1^2} \ge \lambda_{11}E[(X_1)^2] + \sum_{i=2}^{m} \lambda_{ii}k_i^2E[(X_i)^2]$$

$$+\sum_{j=2}^{m} \lambda_{1j}E[(X_1+k_jX_j)^2] + \sum_{2\le i<j\le m} \lambda_{ij}E[(k_iX_i+k_jX_j)^2]$$

$$+\lambda_{1...m}E[(X_1+k_2X_2+..+k_mX_m)^2] > 0. \tag{4.3.17}$$

From (4.3.17) it results that the function $h(r_1)$ is convex in $r_1 \in (0, r_1^0)$, so that if h has a turning point, then the function attains its minimum at that turning point.

Consequently, since k_2,...,k_m may be any non-negative constants, along every ray from the origin into $[0,\infty)^m$, $g(r_1, r_2, ..., r_m)$ is a continuous, decreasing function at zero, it is convex and such that $g(0, ..., 0) = 0$. Therefore, $g(r_1, ..., r_m) < 0$

for all $(r_1, ..., r_m) \in M$ from an arbitrary neighborhood of $(0, ..., 0)$, which together with the continuity and the hypothesis completes the proof. $\square$

Remark 4.3.1. *Following the proof of the above theorem, we point out that for given $k_2, ..., k_m$ non-negative constants, such that*
$\sup_{(r_1,k_2r_1,..,k_mr_1)\in M} g(r_1, k_2r_1, .., k_mr_1) > 0$*, the equation*

$$g(r_1, k_2r_1, ..., k_mr_1) = 0$$

has a unique solution in $(0, r_1^0)$.

We conclude this section with the following remark regarding some situations when the assumption $\sup_{(r_1,...,r_m)\in M} g(r_1, ..., r_m) > 0$ from Theorem 4.3.2 is satisfied.

Remark 4.3.2. *In the case when $\lambda_{ii} \neq 0$, for $i = 1, ..., m$, and $M = [0, r_1^0) \times ... \times [0, r_m^0)$, the condition*

$$\sup_{(r_1,...,r_m)\in M} g(r_1, ..., r_m) > 0 \tag{4.3.18}$$

imposed in Theorem 4.3.2 is satisfied since the following property holds:

$$\lim_{r_1\uparrow r_1^0,...,r_m\uparrow r_m^0} g(r_1, ..., r_m) = \infty. \tag{4.3.19}$$

Indeed, for $r_1 \in [0, r_1^0)$, ...,$r_m \in [0, r_m^0)$,

$$\sum_{1\le i<j\le m} \lambda_{ij} \left[M_{X_i,X_j}(r_i, r_j) - 1\right] \ge 0,$$
$$\lambda_{1...m}[M_{X_1,...,X_m}(r_1, ..., r_m) - 1] \ge 0,$$

which lead to

$$g(r_1, ..., r_m) \ge \sum_{i=1}^{m} \left[-c_ir_i + \lambda_{ii} \left(M_{X_i}(r_i) - 1\right)\right]. \tag{4.3.20}$$

Now, by using (4.3.20) and

$$\lim_{r_i\uparrow r_i^0} \left[-c_ir_i + \lambda_{ii} \left(M_{X_i}(r_i) - 1\right)\right] = \infty, \tag{4.3.21}$$

for each $i = 1, ..., m$ and $0 < r_i^0 \le \infty$, (4.3.19) follows. The property (4.3.21) is actually the result from the univariate case: $\lim_{r\uparrow r^0} g(r) = \infty$, $g(r)$ being defined by (2.3.4), and the proof can be found in Dickson (2005).

The risk model (4.2.1) with the presence of the independent Poisson processes $N_{ii}(t)$, $i = 1, ..., m$, suggested by the condition $\lambda_{ii} \neq 0$, for $i = 1, ..., m$, is suitable for real situations, in the sense that it is natural to consider first the underlying *independent risks specific to each class of business and then to take into consideration the risks generated by common shocks.*

4.4. An upper bound for the infinite-time ruin probability of type ψ_{sim}

With the aid of the martingale established in Theorem 4.3.1 an upper bound for the ruin probability ψ_{sim} is obtained in the following theorem. The proof follows along the same lines as Lundberg's inequality in the classical risk model, proof presented in Subsection 2.3.4.

Theorem 4.4.1. *Consider the risk model* (4.2.1) *and* $g(r_1, ..., r_m)$ *defined by* (4.3.15). *If* $\sup_{(r_1,...,r_m)\in M} g(r_1, ..., r_m) > 0$, *then*

$$\psi_{sim}(u_1, ..., u_m) \leq \inf_{(r_1,...,r_m)\in S} e^{-\sum_{i=1}^{m} r_i u_i},$$

where $S = \{(r_1, ..., r_m) \in M \mid g(r_1, ..., r_m) = 0\}$.

Proof. In view of the proof of Theorem 4.3.2, we have that the set denoted by

$$M_1 = \{(r_1, ..., r_m) \in M \mid g(r_1, ..., r_m) \leq 0\}$$

is non-empty.

Let $(r_1, ..., r_m)$ be an arbitrary element of M and a fixed $t > 0$. Then, the stopping time $\tau_{sim} \wedge t = \min(\tau_{sim}, t)$ is bounded by t and the martingale stopping property, given by Proposition 2.1.4, can be applied for the martingale $Z(t)$ obtained in Theorem 4.3.1. Thus, we have

$$e^{-r_1u_1-...-r_mu_m} = E[Z(0)] = E[e^{-r_1U_1(\tau_{sim}\wedge t)-...-r_mU_m(\tau_{sim}\wedge t)-g(r_1,...,r_m)(\tau_{sim}\wedge t)}]$$

$$= E[e^{-r_1U_1(\tau_{sim})-...-r_mU_m(\tau_{sim})-g(r_1,...,r_m)\tau_{sim}}; \tau_{sim} \leq t]$$

$$+E[e^{-r_1U_1(t)-...-r_mU_m(t)-g(r_1,...,r_m)t}; \tau_{sim} > t]$$

$$\geq E[e^{-r_1U_1(\tau_{sim})-...-r_mU_m(\tau_{sim})-g(r_1,...,r_m)\tau_{sim}}|\tau_{sim} \leq t]\ P(\tau_{sim} \leq t)$$

$$\geq E[e^{-g(r_1,...,r_m)\tau_{sim}}|\tau_{sim} \leq t]\ P(\tau_{sim} \leq t),$$

since $U_i(\tau_{sim}) < 0$ on $(\tau_{sim} < \infty)$, for all $i = 1, 2, ..., m$. Thus,

$$P(\tau_{sim} \leq t) \leq e^{-r_1 u_1 - ... - r_m u_m} \sup_{l \in [0,t]} e^{lg(r_1,...,r_m)}. \tag{4.4.1}$$

Letting $t \to \infty$ in (4.4.1) yields

$$\psi_{sim}(u_1, ..., u_m) \leq e^{-r_1 u_1 - ... - r_m u_m} \sup_{l \geq 0} e^{lg(r_1,...,r_m)}. \tag{4.4.2}$$

Under the restriction $\sup_{l \geq 0} e^{lg(r_1,...,r_m)} < \infty$, it follows from (4.4.2) that

$$\psi_{sim}(u_1, ..., u_m) \leq \inf_{(r_1,...,r_m) \in M_1} e^{-r_1 u_1 - ... - r_m u_m}.$$

Now, for given $k_2 \geq 0$,...,$k_m \geq 0$, let $(s, k_2 s, ..., k_m s)$ be a solution of the equation $g(r_1, ..., r_m) = 0$. Following the proof of Theorem 4.3.2, since $h(r_1) = g(r_1, k_2 r_1, ..., k_m r_1)$ is convex on $(0, r_1^0)$, decreasing at zero, $h(0) = 0$ and $h(s) = 0$, it results that $g(r_1, k_2 r_1, ..., k_m r_1) < 0$ for $0 < r_1 < s$ and $g(r_1, k_2 r_1, ..., k_m r_1) > 0$ for $r_1 > s$. Therefore, if we denote $R_1 = \{(r_1, k_2 r_1, ..., k_m r_1) \mid 0 < r_1 < s\}$, then $\{(s, k_2 s, ..., k_m s)\} \cup R_1 \subseteq M_1$ and for all $(r_1, k_2 r_1, ..., k_m r_1) \in R_1$ the following inequality holds

$$e^{-su_1 - k_2 s u_2 - .. - k_m s u_m} \leq e^{-r_1 u_1 - k_2 r_1 u_2 - .. - k_m r_1 u_m},$$

which yields

$$\psi_{sim}(u_1, ..., u_m) \leq \inf_{(r_1,...,r_m) \in S} e^{-\sum\limits_{i=1}^{m} r_i u_i}.$$

This completes the proof. □

Remark 4.4.1. *Note that the technique applied in the proof of Theorem* 4.4.1 *is not successful when considering the ruin probability* ψ_{and} *since, for example, in establishing the inequality* (4.4.1) *we used the property that*

$$U_i(\tau_{sim}) < 0 \textit{ on } (\tau_{sim} < \infty),$$

for all $i = 1, 2, ..., m$, *which is not valid on neither* $(\tau_{and} < \infty)$ *nor* $(\tau_{or} < \infty)$.

In the case where $N_{ij}(t) \equiv 0$ for $1 \leq i \leq j \leq m$, the risk model (4.2.1) is based only on common shocks given by the Poisson process $\{N_{1...m}(t), t \geq 0\}$ that affect all classes and the result given by Theorem 4.4.1 becomes the result of Proposition 2.4.10, which can be found in Asmussen and Albrecher (2010).

4.5. Expression for the ruin probability of type ψ_{or}

In the following proposition, using the martingale obtained in Theorem 4.3.1, we establish an expression for the probability that ruin occurs at least in one class of business in terms of the solutions of equation $g(r_1, ..., r_m) = 0$ with $g(r_1, ..., r_m)$ defined by (4.3.15).

Theorem 4.5.1. *Consider the risk model* (4.2.1) *and* $g(r_1, ..., r_m)$ *defined by* (4.3.15) *such that* $\sup_{(r_1,...,r_m)\in M} g(r_1, ..., r_m) > 0$.
If $(r_1, .., r_m) \in S$, *where* $S = \{(r_1, ..., r_m) \in M \mid g(r_1, ..., r_m) = 0\}$, *then*

$$\psi_{or}(u_1, ..., u_m) = \frac{e^{-r_1u_1-...-r_mu_m}}{E[e^{-r_1U_1(\tau_{or})-...-r_mU_m(\tau_{or})} \mid \tau_{or} < \infty]}. \tag{4.5.1}$$

Proof. Since $(r_1, ..., r_m)$ is an element from S, we have $g(r_1, ..., r_m) = 0$ and consequently, the martingale obtained in Theorem 4.3.1 becomes
$Z(t) = e^{-r_1U_1(t)-...-r_mU_m(t)}$. For a fixed $t > 0$, consider $\tau_{or} \wedge t = \min(\tau_{or}, t)$, which is a stopping time bounded by t, and therefore, the martingale stopping property, given by Proposition 2.1.4, can be applied for $Z(t)$ as follows:

$$e^{-r_1u_1-...-r_mu_m} = E[Z(0)] = E[e^{-r_1U_1(\tau_{or}\wedge t)-...-r_mU_m(\tau_{or}\wedge t)}]$$

$$= E[e^{-r_1U_1(\tau_{or})-...-r_mU_m(\tau_{or})}; \tau_{or} \le t] + E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t]. \tag{4.5.2}$$

As $t \to \infty$,

$$E[e^{-r_1U_1(\tau_{or})-...-r_mU_m(\tau_{or})}; \tau_{or} \le t] \to E[e^{-r_1U_1(\tau_{or})-...-r_mU_m(\tau_{or})}; \tau_{or} < \infty]$$

$$= E[e^{-r_1U_1(\tau_{or})-...-r_mU_m(\tau_{or})} \mid \tau_{or} < \infty]P(\tau_{or} < \infty). \tag{4.5.3}$$

Now, we prove that the second term in (4.5.2) vanishes as $t \to \infty$. By the net profit condition for each class of business, it follows that for all $i = 1, ..., m$, $U_i(t) \to \infty$ almost surely as $t \to \infty$, which imply

$$P(U_i(t) \le \epsilon_i) \to 0 \text{ as } t \to \infty, \tag{4.5.4}$$

for any arbitrary large $\epsilon_i > 0$. Let ϵ_i, $i = 1, ..., m$ be arbitrary large strictly positive numbers. Then

$$E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t]$$

$$= E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t, U_i(t) \leq \epsilon_i, i = 1, ..., m]$$
$$+E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t, U_i(t) > \epsilon_i, i = 1, ..., m]$$

$$+\sum_{k=1}^{m-1} \sum_{1\leq i_1<i_2<...<i_k\leq m} E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t, U_{i_1}(t) \leq \epsilon_{i_1}, .., U_{i_k}(t) \leq \epsilon_{i_k},$$

$$U_j > \epsilon_j, j \neq i_1, ..., i_k, j = 1, ..., m]$$

$$\leq P\left(U_i(t) \leq \epsilon_i, i = 1, ..., m\right) + e^{-r_1\epsilon_1-...-r_m\epsilon_m}$$

$$+\sum_{k=1}^{m-1} \sum_{1\leq i_1<i_2<...<i_k\leq m} e^{-\sum\limits_{j=1;j\neq i_1,..,i_k}^{m} r_j\epsilon_j} P\left(U_{i_1}(t) \leq \epsilon_{i_1}, ..., U_{i_k}(t) \leq \epsilon_{i_k}\right), \qquad (4.5.5)$$

where we used the fact that for all $i = 1, ..., m$, $U_i(t) \geq 0$ when $t < \tau_{or}$ leading to $e^{-r_iU_i(t)} \leq 1$ and that for those i such that $U_i(t) > \epsilon_i$ then $e^{-r_iU_i(t)} \leq e^{-r_i\epsilon_i}$. Letting $t \to \infty$ in (4.5.5), it follows that

$$\lim_{t\to\infty} E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t] \leq e^{-r_1\epsilon_1-...-r_m\epsilon_m},$$

due to conditions (4.5.4), and further as $\epsilon_i \to \infty$, $i = 1, ..., m$, we get

$$\lim_{t\to\infty} E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t] = 0.$$

Using this result together with the ones given by (4.5.2) and (4.5.3) establishes relation (4.5.1). □

Remark 4.5.1. *Note that the technique used in obtaining the result* (4.5.1) *can not be applied to the ruin time of type* τ_{and} *since the property*

$$U_i(t) \geq 0 \text{ on } (t < \tau_{or}) \text{ for all } i = 1, .., m,$$

used in showing that $\lim_{t\to\infty} E[e^{-r_1U_1(t)-...-r_mU_m(t)}; \tau_{or} > t] = 0$*, does not hold in the case of* τ_{and}*; it also does not hold for* τ_{sim}*.*

When $m = 1$, result (4.5.1) becomes the well-known expression of the ruin probability for the classical risk model, given by Proposition 2.3.2. Therefore, even though the denominator in (4.5.1) is not easy to evaluate, from a theoretical point of view it represents a generalization of the result from the univariate case.

4.6. Asymptotic behavior of the finite-time ruin probability

From a more practical point of view, the finite-time ruin probability $\psi(u,t)$ introduced by Definition 2.2.3, where t is related to the planning horizon of the company, may be regarded as more interesting than the infinite time ruin probability $\psi(u)$ because most insurance managers will closely follow the development of the risk business and increase the premium if the risk behaves badly.

As we mentioned in Subsection 2.2.2, in recent years, many researchers have shown great interest in studying heavy-tailed distributions which are more realistic in modeling claim amounts (large claims), especially those from general insurance; for more details on applications of these distributions to insurance and finance see, for example, Bingham et al. (1987) and Embrechts et al. (1997). Also, we saw that a different asymptotic behavior of ruin probabilities is observed depending on whether light-tailed or heavy-tailed claim size distributions are considered. If the claim size distribution is light-tailed, the ruin probabilities will turn out to be typically exponentially bounded as the initial capital becomes large. However, when the claim size distribution has a heavy tail, then one single large claim may be responsible for the ultimate ruin of the portfolio.

These facts constitute the reasons for which we consider studying the behavior of the ruin probability $\psi_{sim}(u_1,...,u_m,t)$ as the initial surpluses u_1,...,u_m become large for a fixed time $t>0$, in both situations where the claim sizes are heavy-tailed dependent and heavy-tailed independent. This section is devoted to this investigation.

First, we establish some preliminary results which will be used in this section.

As in the univariate model discussed in Section 2.2, consider the multivariate risk model defined by (4.2.1) without having satisfied the net profit conditions. The finite-time ruin probability $\psi_{sim}(u_1,...,u_m,t)$ is given by

$$\psi_{sim}(u_1,...,u_m,t) = P\left(\tau_{sim} \le t | U_1(0)=u_1,...,U_m(0)=u_m\right)$$

$$= P\left(\sum_{k=1}^{N_i(s)} X_{ik} - c_i\, s > u_i,\ i=1,2,...,m,\ \text{for some } 0 < s \le t\right). \qquad (4.6.1)$$

Formula (4.6.1) together with the following inequalities obtained for each $s \in (0,t]$

$$\sum_{k=1}^{N_i(s)} X_{ik} - c_i\, t \le \sum_{k=1}^{N_i(s)} X_{ik} - c_i\, s \le \sum_{k=1}^{N_i(t)} X_{ik},\ \ i = 1, 2, .., m,$$

lead to the following two relations

$$\psi_{sim}(u_1, ..., u_m, t) \le P\left(\sum_{k=1}^{N_i(t)} X_{ik} > u_i,\ \ i = 1, 2, .., m\right), \tag{4.6.2}$$

and

$$\begin{aligned}\psi_{sim}(u_1, ..., u_m, t) &\ge P\left(\sum_{k=1}^{N_i(s)} X_{ik} - c_i\, t > u_i,\ \ i = 1, 2, ..., m,\ \text{ for some } 0 < s \le t\right)\\ &= P\left(\sum_{k=1}^{N_i(t)} X_{ik} - c_i\, t > u_i,\ \ i = 1, 2, ..., m\right). \end{aligned}\tag{4.6.3}$$

The first subsection aims at providing an asymptotic upper bound for the finite-time ruin probability $\psi_{sim}(u_1, ..., u_m, t)$ in the situation of dependent heavy-tailed claim sizes.

The second subsection deals with the case of independent heavy-tailed claim sizes and an asymptotic result is obtained, which is an extension of the result from the classical risk model, given by Proposition 2.3.15. For this case, a similar result for the multivariate risk model (4.2.1) when $N_{ij}(t) \equiv 0$, for $1 \le i \le j \le m$, was obtained by Asmussen and Albrecher (2010).

4.6.1. Dependent claims

Since we are interested in considering dependent heavy-tailed claim sizes, we first present the following result formulated by Ko and Tang (2008) concerning the asymptotic tail probability of sums of dependent and heavy-tailed nonnegative random variables. For the multivariate case, they considered the following assumption:

Assumption 1. Let X_1,...,X_n be n $(n \ge 2)$ random variables with distributions F_1,...,F_n concentrated on $[0, \infty)$, respectively. Assume that there exists some large $x_0 > 0$ such that, for every $j = 2, .., n$, the relation

$$\frac{P(X_1 + .. + X_{j-1} > x - t \mid X_j = t)}{P(X_1 + .. + X_{j-1} > x - t)} = O(1)$$

holds uniformly for all $t \in [x_0, x]$, meaning that

$$\limsup_{x\to\infty} \sup_{x_0 \le t \le x} \frac{P(X_1 + .. + X_{j-1} > x - t \mid X_j = t)}{P(X_1 + .. + X_{j-1} > x - t)} < \infty.$$

The following proposition is formulated as Theorem 3.1 in Ko and Tang (2008).

Proposition 4.6.1. *Let $X_1,\ldots,X_n$ be n ($n \ge 2$) random variables with distributions $F_1,\ldots,F_n$ concentrated on $[0,\infty)$, respectively, such that Assumption 1 holds for all $j = 2,...,n$. Then the relations*

$$P(X_1 + ... + X_n > x) \sim P(\max\{X_1,...,X_n\} > x) \sim \sum_{k=1}^{n} \overline{F_k}(x) \tag{4.6.4}$$

hold for each of the following two cases:

(i) $F_k \in \mathcal{S}$ for all $k = 1,..,n$, and either $\overline{F_i}(x) = O(\overline{F_j}(x))$ or $\overline{F_j}(x) = O(\overline{F_i}(x))$ for all $i,j = 1,...,n$;

(ii) $F_k \in \mathcal{D} \cap \mathcal{L}$ for all $k = 1,...,n$.

Remark 4.6.1. *A by-product of this result is that the distribution of $X_1 + ... + X_n$ belongs to $\mathcal{S}$ for case (i) and belongs to $\mathcal{D} \cap \mathcal{L}$ for case (ii).*

By using Proposition 4.6.1, we derive an asymptotic upper bound for the finite-time ruin probability $\psi_{sim}(u_1,...,u_m,t)$, as follows.

Theorem 4.6.1. *Consider the risk model given by (4.2.1) and suppose that the claim sizes $X_1,\ldots,X_m$ ($m \ge 2$) follow a dependence structure characterized by Assumption 1, with their distributions $F_1,\ldots,F_m$ satisfying the conditions of cases (i) or (ii) of Proposition 4.6.1.*

Then, as $u_1 \to \infty,\ldots,u_m \to \infty$, the following inequality holds

$$\psi_{sim}(u_1,...,u_m,t) \preceq E[\max\{N_1(t),...,N_m(t)\}] \sum_{i=1}^{m} \overline{F_i}(u_1 + ... + u_m),$$

where $f(x) \preceq g(x)$ means that $\limsup f(x)/g(x) \le 1$ as $x \to \infty$.

Proof. Let us denote $p_{n_1,\ldots,n_m} = P\left(N_1(t) = n_1,...,N_m(t) = n_m\right)$ and $Y_k = X_{1k} + ... + X_{mk}$ for $k \ge 1$. By inequality (4.6.2), we obtain that

$$\psi_{sim}(u_1,...,u_m,t) \le \sum_{n_1,\ldots,n_m=0}^{\infty} p_{n_1,\ldots,n_m} P\left(\sum_{k=1}^{n_i} X_{ik} > u_i,\ i = 1,...,m\right)$$

$$\leq \sum_{n_1,...,n_m=0}^{\infty} p_{n_1,...,n_m} P\left(\sum_{k=1}^{n_1} X_{1k} + ... + \sum_{k=1}^{n_m} X_{mk} > u_1 + ... + u_m\right)$$

$$\leq \sum_{n_1,...,n_m=0}^{\infty} p_{n_1,...,n_m} P\left(\sum_{k=1}^{\max\{n_1,...,n_m\}} Y_k > u_1 + ... + u_m\right), \tag{4.6.5}$$

where for the first inequality, we applied the inclusion of events

$$\left(\sum_{k=1}^{n_i} X_{ik} > u_i,\ i = 1, ..., m\right) \subseteq \left(\sum_{k=1}^{n_1} X_{1k} + ... + \sum_{k=1}^{n_m} X_{mk} > u_1 + ... + u_m\right),$$

and for the second inequality, since the X_{ik}'s are positive, we applied the inequalities

$$\sum_{k=1}^{n_i} X_{ik} \leq \sum_{k=1}^{\max\{n_1,...,n_m\}} X_{ik} \text{ for all } i = 1, ..., m,$$

and hence,

$$\left(\sum_{k=1}^{n_1} X_{1k} + ... + \sum_{k=1}^{n_m} X_{mk} > u_1 + ... + u_m\right) \subseteq \left(\sum_{k=1}^{\max\{n_1,...,n_m\}} Y_k > u_1 + ... + u_m\right).$$

Using Remark 4.6.1, for each $k \geq 1$, the distribution of $Y_k = X_{1k} + ... + X_{mk}$ belongs to $\mathcal{S}$ in case (i) and to $\mathcal{D} \cap \mathcal{L}$ in case (ii), both cases being specified in Proposition 4.6.1. Since $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S}$ (see Proposition 2.2.8), we obtain that in both cases the distribution of $Y_k = X_{1k} + ... + X_{mk}$ belongs to $\mathcal{S}$. Moreover, $\{Y_k\}_{k\geq 1}$ are independent and identically distributed random variables with common distribution function $F_{X_1+...+X_m}$. Consequently, using relation (2.2.8) from Lemma 2.2.1, it follows that for an arbitrarily fixed $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that the inequality

$$P\left(\sum_{k=1}^{\max\{n_1,..,n_m\}} Y_k > u_1 + ... + u_m\right) \leq C(\epsilon)(1+\epsilon)^{\max\{n_1,..,n_m\}} P\left(Y_k > u_1 + ... + u_m\right) \tag{4.6.6}$$

holds for all $n_1, ..., n_m = 1, 2, ...$ and $u_1, ..., u_m \geq 0$. Consequently,

$$\sum_{n_1,...,n_m=0}^{\infty} p_{n_1,...,n_m} P\left(\sum_{k=1}^{\max\{n_1,...,n_m\}} Y_k > u_1 + ... + u_m\right)$$

$$\leq C(\epsilon) P\left(Y_k > u_1 + ... + u_m\right) E[(1+\epsilon)^{\max\{N_1(t),...,N_m(t)\}}],$$

and since $\{N_{ij}(t), t \geq 0\}$, for $1 \leq i, j \leq m$, and $\{N_{1...m}(t), t \geq 0\}$ are all mutually independent Poisson processes,

$$E[(1+\epsilon)^{\max\{N_1(t),...,N_m(t)\}}] \leq E[(1+\epsilon)^{N_1(t)+...+N_m(t)}]$$

$$= \prod_{i=1}^{m} E[(1+\epsilon)^{N_{ii}(t)}] \prod_{1 \leq i < j \leq m}^{m} E[(1+\epsilon)^{2N_{ij}(t)}] E[(1+\epsilon)^{mN_{12..m}(t)}] < \infty. \quad (4.6.7)$$

Thus, in view of (4.6.6) and (4.6.7), the dominated convergence theorem can be applied to (4.6.5) along with the definition of subexponentiality (2.2.5) for Y_k leading to the following result

$$\limsup \frac{\psi_{sim}(u_1, ..., u_m, t)}{P\left(Y_k > u_1 + ... + u_m\right)} \leq \sum_{n_1,...,n_m=0}^{\infty} p_{n_1,...,n_m} \max\{n_1, ..., n_m\}, \quad (4.6.8)$$

as $u_1 \to \infty$,...,$u_m \to \infty$.

Now, applying Proposition 4.6.1 to $Y_k = X_{1k} + ... + X_{mk}$, we have that

$$\lim \frac{P\left(Y_k > u_1 + ... + u_m\right)}{\sum_{i=1}^{m} \overline{F_i}(u_1 + ... + u_m)} = 1 \text{ as } u_1 \to \infty, ..., u_m \to \infty,$$

which together with relation (4.6.8) completes the proof. $\square$

Proposition 4.6.2 shows that an upper bound of the ruin probability $\psi_{sim}(u_1, ..., u_m, t)$ is determined by $E[\max\{N_1(t), ..., N_m(t)\}] \sum_{i=1}^{m} \overline{F_i}(u_1 + ... + u_m)$ as the initial surpluses u_1,...,u_m increase.

We conclude this subsection by showing how the Assumption 1 is satisfied for the risk model (4.2.1). For this, we present the following example of dependence structure of the claims proposed by Ko and Tang (2008).

Example 4.6.1. *Let the random variables X_1,...,X_n be dependent according to a multivariate copula function $C(u_1, ..., u_n)$ and let their distributions F_1,...,F_n be absolutely continuous and satisfy the conditions of cases (i) or (ii) of Proposition* 4.6.1. *Recall from Section* 2.4.1, *the notion of multivariate copula introduced by Definition* 2.4.2. *Assume that the copula density exists:*

$$C_{1...n}(u_1, ..., u_n) = \frac{\partial^n}{\partial u_1 ... \partial u_n} C(u_1, ..., u_n).$$

If, for every nonempty subset I of $\{1, ..., n\}$, the marginal copula density $C_I(u_I : i \in I)$ is bounded in a neighborhood of the ultimate vertex (whose coordinates are all 1*), then Assumption* 1 *is fulfilled.*

A copula whose joint copula density is uniformly bounded in the whole domain satisfies the above requirements. For example, copulas in the Frank family of the form

$$C(u_1, ..., u_n; \theta) = -\frac{1}{\theta}\left(1 + \frac{(e^{-\theta u_1} - 1)...(e^{-\theta u_n} - 1)}{(e^{-\theta} - 1)^{n-1}}\right), \quad \theta > 0,$$

as well as copulas in the Clayton family introduced by Example 2.4.4, *belong to this category.*

4.6.2. Independent claims

In a similar manner as in obtaining the result for the classical risk model, given by Proposition 2.3.15, we derive an asymptotic result for the finite-time ruin probability $\psi_{sim}(u_1, ..., u_m, t)$, for a fixed time $t > 0$, in the case of independent heavy-tailed claims. This result is described by the following proposition.

Theorem 4.6.2. *Consider the multivariate risk model defined by* (4.2.1) *with the assumption that the claim sizes* $X_1,...,X_m$ $(m \geq 2)$ *are independent and subexponentially distributed. Then, as* $u_1 \to \infty,...,u_m \to \infty$*, we have*

$$\psi_{sim}(u_1, ..., u_m, t) \sim E[N_1(t) \times ... \times N_m(t)]\overline{F_1}(u_1)...\overline{F_m}(u_m).$$

Proof. In view of the inequalities (4.6.2) and (4.6.3), it is sufficient to establish the following two asymptotic results:

$$P\left(\sum_{k=1}^{N_i(t)} X_{ik} > u_i,\ i = 1, 2, .., m\right) \sim E[N_1(t) \times ... \times N_m(t)]\overline{F_1}(u_1)...\overline{F_m}(u_m), \tag{4.6.9}$$

and

$$P\left(\sum_{k=1}^{N_i(t)} X_{ik} > u_i + c_i\, t\right) \sim E[N_1(t) \times ... \times N_m(t)]\overline{F_1}(u_1)...\overline{F_m}(u_m). \tag{4.6.10}$$

We start by showing (4.6.9). In view of the independence assumption, we obtain

$$P\left(\sum_{k=1}^{N_i(t)} X_{ik} > u_i,\ i = 1, 2, .., m\right) = \sum_{n_1,..,n_m=0}^{\infty} p_{n_1,..,n_m} \prod_{i=1}^{m} P\left(\sum_{k=1}^{n_i} X_{ik} > u_i\right), \tag{4.6.11}$$

For each $i = 1, 2, .., m$, the random variables X_{ik}, $k \geq 1$, are independent and subexponentially distributed, and therefore, as $u_i \to \infty$, by (2.2.5) we have

$$P\left(\sum_{k=1}^{n_i} X_{ik} > u_i\right) \sim n_i \overline{F_i}(u_i). \tag{4.6.12}$$

Also, by (2.2.8), we have that for every $\epsilon > 0$, there exists a constant $C_i(\epsilon) > 0$ such that the inequality

$$P\left(\sum_{k=1}^{n_i} X_{ik} > u_i\right) \leq C_i(\epsilon)(1+\epsilon)^{n_i} P\left(X_i > u_i\right)$$

holds for all $n_i = 1, 2, ...$ and $u_i \geq 0$. Thus, the right-hand side of (4.6.11) satisfies

$$\sum_{n_1,..,n_m=0}^{\infty} p_{n_1,..,n_m} \prod_{i=1}^{m} P\left(\sum_{k=1}^{n_i} X_{ik} > u_i\right) \leq \prod_{i=1}^{m} C_i(\epsilon) P\left(X_i > u_i\right) E[(1+\epsilon)^{\sum\limits_{i=1}^{m} N_i(t)}] < \infty. \tag{4.6.13}$$

By substituting (4.6.7) into (4.6.13), dominated convergence theorem can be applied in relation (4.6.11) along with the definition of subexponentiality (4.6.12) for X_i and we conclude that

$$P\left(\sum_{k=1}^{N_i(t)} X_{ik} > u_i,\ i = 1, 2, .., m\right) \sim \sum_{n_1,..,n_m=0}^{\infty} p_{n_1,..,n_m} n_1..n_m \overline{F_1}(u_1)...\overline{F_m}(u_m)$$

as $u_i \to \infty$, $i = 1, 2, .., m$. Hence, result (4.6.9) is obtained.

We proceed in a similar manner to obtain result (4.6.10). Thus, the independence of the claim sizes yields

$$P\left(\sum_{k=1}^{N_i(t)} X_{ik} > u_i + c_i\, t,\ i = 1, 2, .., m\right)$$

$$= \sum_{n_1,..,n_m=0}^{\infty} p_{n_1,..,n_m} \prod_{i=1}^{m} P\left(\sum_{k=1}^{n_i} X_{ik} > u_i + c_i\, t\right). \tag{4.6.14}$$

For each $i = 1, 2, .., m$, the random variables X_{ik}, $k \geq 1$, are independent and subexponentially distributed, and therefore, by (2.2.5), as $u_i \to \infty$,

$$P\left(\sum_{k=1}^{n_i} X_{ik} > u_i + c_i\, t\right) \sim n_i \overline{F_i}(u_i + c_i\, t) \sim n_i \overline{F_i}(u_i), \tag{4.6.15}$$

where at the second step, the inequality $\mathcal{S} \subset \mathcal{L}$ (see Proposition 2.2.8) was used. Again, using the dominated convergence theorem, as was done for proving (4.6.9), along with using (4.6.15) into relation (4.6.14) yield the desired result (4.6.10). $\square$

Theorem 4.6.2 shows that the ruin probability $\psi_{sim}(u_1, ..., u_m, t)$ is asymptotically determined by $E[N_1(t) \times ... \times N_m(t)]\overline{F_1}(u_1)...\overline{F_m}(u_m)$. In regard with the expectation $E[N_1(t) \times ... \times N_m(t)]$, for fixed $t > 0$, it can be computed with the aid of the probability generating function (p.g.f.) of the multivariate Poisson distribution, namely

$$E[N_1(t) \times ... \times N_m(t)] = \frac{\partial^m P_{N_1(t),..,N_m(t)}(z_1,...,z_m)}{\partial z_1...\partial z_m}|_{z_1=1,...,z_m=1},$$

where the p.g.f., denoted by $P_{N_1(t),..,N_m(t)}(z_1, ..., z_m)$, is expressed as

$$P_{N_1(t),..,N_m(t)}(z_1, ..., z_m) = E[z_1^{N_1(t)}...z_m^{N_m(t)}]$$

$$= \prod_{i=1}^{m} P_{N_{ii}(t)}(z_i) \prod_{1 \le i < j \le m} P_{N_{ij}(t)}(z_i z_j)\ P_{N_{1...m}(t)}(z_1...z_m),$$

since $\{N_{ij}(t), t \geq 0\}$, for $1 \leq i, j \leq m$, and $\{N_{1...m}(t), t \geq 0\}$ are all mutually independent Poisson processes. According to property (4) of Proposition 2.2.4, if $N(t) \sim \text{Poisson}(\lambda t)$, then the probability generating function is given by $P_{N(t)}(z) = e^{\lambda t(z-1)}$ and hence, $E[N(t)] = P'_{N(t)}(1) = \lambda t$.

It is worth considering some specific cases of Theorem 4.6.2. For instance, for the multivariate model (4.2.1) with $N_{ij}(t) \equiv 0$ for $1 \leq i \leq j \leq m$ and $N_{1...m}(t) \equiv N(t) \sim \text{Poisson}(\lambda t)$, the asymptotic result of Theorem 4.6.2 becomes

$$\psi_{sim}(u_1, ..., u_m, t) \sim E[(N(t))^m]\overline{F_1}(u_1)...\overline{F_m}(u_m), \tag{4.6.16}$$

as $u_1 \to \infty$,...,$u_m \to \infty$.

Result (4.6.16) was established by Asmussen and Albrecher (2010), under the assumption that the claim number process $\{N(t), t \geq 0\}$ is a counting process such that $E[z^{N(t)}] < \infty$.

For $m = 1$, the asymptotic result of Theorem 4.6.2 is the asymptotic result established in the classical risk model given by relation (2.3.20).

4.7. Multivariate risk model perturbed by diffusion

Dufresne and Gerber (1991) derived an upper bound of Lundberg type for the ruin probability $\psi(u)$ associated to the classical risk model perturbed by a Brownian motion defined by (2.3.32), result which is illustrated by relation (2.3.34). Later on, Li, Liu and Tang (2007) obtained an upper bound for the ruin probability $\psi_{sim}(u_1, u_2)$ associated to the bivariate risk model perturbed by a bidimensional Brownian motion with constant correlation coefficient $r \in [-1, 1]$ defined by (2.4.19), where the claim counting processes are assumed to be the same Poisson process. This result is illustrated by Proposition 2.4.8.

Inspired by these papers, we embrace the idea of adding a diffusion process to the risk model (4.2.1) and this way, we extend the results from univariate and bivariate models to the multivariate models. Therefore, this multivariate perturbed risk model is obtained by adding a correlated m-dimensional Brownian motion to the risk model (4.2.1), where the claims arrivals follow a Poisson model with common shocks. Mathematically, the m surplus processes $U_i(t)$ defined by (4.2.1) are extended to

$$V_i(t) = U_i(t) + \sigma_i B_i(t), \ i = 1, ..., m, \tag{4.7.1}$$

where $\mathbf{B}(t) = (B_1(t), ..., B_m(t))$ denotes a standard correlated m-dimensional Brownian motion with constant correlation coefficients $\rho_{ii} = 1$, $\rho_{ij} = \rho_{ji} \in [-1, 1]$, $i, j = 1, ..., m$, and $\sigma_i \geq 0$ are the diffusion volatility coefficients of $B_i(t)$. Furthermore, it is assumed that $\{N_{ij}(t), t \geq 0\}$, for $1 \leq i \leq j \leq m$, $\{N_{1...m}(t), t \geq 0\}$, $\{(X_{1k}, ..., X_{mk})\}_{k \geq 1}$ and $\{\mathbf{B}(t), t \geq 0\}$ are all mutually independent.

The diffusion term in (4.7.1) accounts for some market uncertainties in the application of risk model (4.2.1) namely, uncertainty of the aggregate claims or of the premium income.

Recall the properties satisfied by the vector process $\mathbf{B}(t) = (B_1(t), ..., B_m(t))$:

1. $\mathbf{B}(0) = \mathbf{0}$, the zero vector in $\mathbb{R}^m$.

2. For all $t \geq 0$ and $h > 0$, $\mathbf{B}(t+h) - \mathbf{B}(t)$ is multivariate normal, with mean zero and variance-covariance matrix $h\mathbf{A}$, where the matrix $\mathbf{A} = (\rho_{ij})_{1 \leq i,j \leq m}$ is a positive symmetric matrix satisfying $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji} \in [-1, 1]$.

3. If $0 \leq r \leq s < t$, then the random variables $\mathbf{B}(t) - \mathbf{B}(s)$ and $\mathbf{B}(r)$ are independent, meaning that each component of the former is independent of each component of the latter.

4. Each component-process $\{B_i(t), t \geq 0\}$ is itself a standard Brownian motion, presented in Subsection 2.3.5.

Since for each $i = 1, ..., m$, $E[B_i(t)] = 0$, the net profit condition (2.2.4) assumed to hold for each class of business is equivalent to (4.2.2).

Note that in the case where $\sigma_i = 0$ for all $i = 1, ..., m$, the risk model (4.7.1) becomes the model without diffusion given by (4.2.1).

Following the discussions from previous sections concerning the ruin concepts from Definition 2.4.1, namely Remarks 4.4.1 and 4.5.1, we conclude that the martingale approach adopted for risk model (4.2.1) can be applied in obtaining results for the ruin probabilities of types ψ_{sim} and ψ_{or} associated to (4.7.1) and hence, these two types of ruin probabilities will be studied this section.

In a similar manner as was done for the unperturbed multivariate risk model (4.2.1), with the aid of Proposition 2.1.5 we derive a martingale process which will be used in establishing a Lundberg-type upper bound for the ruin probability $\psi_{sim}(u_1, ..., u_m)$ associated to the risk model (4.7.1). The bound is similar to the upper bound obtained in Theorem 4.4.1 under the assumption that $\rho_{ij} \in [0, 1]$, except for at most one element, for all $i, j = 1, ..., m$, $i \neq j$ and $m \geq 3$. We show that this restriction of the correlation coefficients is helpful in establishing the convexity property of the function that defines the equation whose solutions appear in the expression of the upper bound. We point out that for the bivariate model studied by Li, Liu and Tang (2007), this convexity property holds if $\rho_{12} \in [-1, 1]$, which is illustrated in this section. We also formulate an upper bound in the case when $\rho_{ij} \in [-1, 1]$ for all $i, j = 1, ..., m$.

As in Section 4.3, assume that the vector process $\{(V_1(t), ..., V_m(t)), t \geq 0\}$ is defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, where $\mathcal{F}_t = \mathcal{F}_t^{V_1} \vee ... \vee \mathcal{F}_t^{V_m}$ and $\mathcal{F}_t^{V_i}$ is the natural filtration of the process $\{V_i(t), t \geq 0\}$, that is,

$$\mathcal{F}_t^{V_i} = \sigma\left(V_i(s) : 0 \leq s \leq t\right), \quad i = 1, ..., m.$$

Using Proposition 2.3.1 and the property that the vector process $\mathbf{B}(t) = (B_1(t), ..., B_m(t))$ has independent and stationary increments yields that the vector-valued process $(V_1(t), ..., V_m(t),)$ has independent and stationary increments and hence, it is a time-homogeneous Markov vector process by Proposition 2.1.1. Next, we proceed to compute the infinitesimal generator of this Markov vector process, which is needed in applying Proposition 2.1.5.

Proposition 4.7.1. *The infinitesimal generator of the homogeneous Markov vector process* $(V_1(t), ..., V_m(t), t)$ *acting on a function* $f(z_1, ..., z_m, t)$ *belonging to its domain is described as*

$$\mathcal{A}f(z_1, ..., z_m, t) = \sum_{i=1}^{m} c_i \frac{\partial f(z_1, ..., z_m, t)}{\partial z_i} + \frac{\partial f(z_1, ..., z_m, t)}{\partial t}$$

$$+\frac{1}{2}\left[\sum_{i=1}^{m} \sigma_i^2 \frac{\partial^2 f(z_1, ..., z_m, t)}{\partial z_i^2} + 2 \sum_{1\leq i<j\leq m} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 f(z_1, ..., z_m, t)}{\partial z_i \partial z_j}\right]$$

$$+\sum_{i=1}^{m} \lambda_{ii} \left[\int_0^\infty f(z_1, ..., z_i - x_i, ..., z_m, t) dF_i(x_i) - f(z_1, ..., z_m, t)\right]$$

$$+\sum_{1\leq i<j\leq m} \lambda_{ij} \left[\int_{[0.\infty)^2} f(z_1, ., z_i - x_i, .., z_j - x_j, ., z_m, t) dF_{i,j}(x_i, x_j) - f(z_1, .., z_m, t)\right]$$

$$+\lambda_{1...m} \left[\int_{[0,\infty)^m} f(z_1 - x_1, ..., z_m - x_m, t) dF(x_1, ..., x_m) - f(z_1, ..., z_m, t)\right],$$

where $f : \mathbb{R}^m \times (0, \infty) \to (0, \infty)$ *is twice differentiable with respect to* z_1,...,z_m, t, *for all* z_1,...,z_m, t.

Proof. By Definition 2.1.6, we have that $\mathcal{A}f(z_1, ..., z_m, t)$ is equal to

$$\lim_{h\downarrow 0} \frac{E[f(V_1(t+h), .., V_m(t+h), t+h) \mid V_i(t) = z_i,\ i = 1, ..., m] - f(z_1, .., z_m, t)}{h}, \tag{4.7.2}$$

where the domain of $\mathcal{A}$ is the set of all measurable functions f for which this limit exists. In view of the relation

$$V_i(t+h) = z_i + c_i h - \sum_{k=N_i(t)+1}^{N_i(t+h)} X_{ik} + \sigma_i (B_i(t+h) - B_i(t)),$$

provided that $V_i(t) = z_i$ for all $i = 1, ..., m$, by similar arguments to those used in the proof of Proposition 4.3.1 regarding the probabilities of occurrence of events modeled by the Poisson process $N(t)$ given by (4.3.2), in a small time interval $(t, t+h]$, the limit (4.7.2) is equal to

$$\lim_{h\downarrow 0}\frac{1}{h}e^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1\dots m}\right]h}E[f(z_1+c_1h+\sigma_1B_1(h),..,z_m+c_mh+\sigma_mB_m(h),t+h)]$$

$$-\lim_{h\downarrow 0}\frac{1}{h}f(z_1,..,z_m,t)$$

$$+\lim_{h\downarrow 0}\sum_{i=1}^{m}\frac{\lambda_{ii}he^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1\dots m}\right]h}}{h}\times\int_0^\infty f(z_1,...,z_i-x_i,...,z_m,t)dF_i(x_i)$$

$$+\lim_{h\downarrow 0}\sum_{1\le i<j\le m}\frac{\lambda_{ij}he^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1\dots m}\right]h}}{h}$$

$$\times\int_{[0,\infty)^2}f(z_1,...,z_i-x_i,...,z_j-x_j,...,z_m,t)dF_{i,j}(x_i,x_j)$$

$$+\lim_{h\downarrow 0}\frac{\lambda_{1\dots m}he^{-\left[\sum\limits_{1\le i\le j\le m}\lambda_{ij}+\lambda_{1\dots m}\right]h}}{h}$$

$$\times\int_{[0,\infty)^m}f(z_1-x_1,...,z_m-x_m,t)dF(x_1,...,x_m)+\lim_{h\downarrow 0}\frac{o(h)}{h}. \tag{4.7.3}$$

Using a Taylor series' expansion for the function $f(z_1, ..., z_m, t)$ and the properties of the m-dimensional standard Brownian motion $(B_1(t), ..., B_m(t))$, we compute the expectation within the first term of the relation (4.7.3) as follows:

$$E[f(z_1+c_1h+\sigma_1B_1(h),...,z_m+c_mh+\sigma_mB_m(h),t+h)]$$

$$=f(z_1,...,z_m,t)+\sum_{i=1}^{m}\frac{\partial f(z_1,...,z_m,t)}{\partial z_i}E[c_ih+\sigma_iB_i(h)]+\frac{\partial f(z_1,...,z_m,t)}{\partial t}h$$

$$+\frac{1}{2}\sum_{i=1}^{m}\frac{\partial^2 f(z_1,...,z_m,t)}{\partial z_i^2}E[(c_ih+\sigma_iB_i(h))^2]$$

$$+\sum_{1\le i<j\le m}\frac{\partial^2 f(z_1,...,z_m,t)}{\partial z_i\partial z_j}E[(c_ih+\sigma_iB_i(h))(c_jh+\sigma_jB_j(h))]+o(h)$$

$$=f(z_1,...,z_m,t)+\sum_{i=1}^{m}\frac{\partial f(z_1,...,z_m,t)}{\partial z_i}c_ih+\frac{\partial f(z_1,...,z_m,t)}{\partial t}h$$

$$+\frac{1}{2}\sum_{i=1}^{m}\frac{\partial^2 f(z_1,...,z_m,t)}{\partial z_i^2}\sigma_i^2 h+\sum_{1\le i<j\le m}\frac{\partial^2 f(z_1,...,z_m,t)}{\partial z_i\partial z_j}\sigma_i\sigma_j\rho_{ij}h+o(h), \quad (4.7.4)$$

since $E[B_i(h)]=0$, $E[(B_i(h))^2]=h$ and $E[B_i(h)B_j(h)]=\rho_{ij}h$. Substituting the result (4.7.4) along with the results of the other limits in (4.7.3), given by (4.3.8), (4.3.9), (4.3.10) and (4.3.11), respectively, establishes the desired result. Note that for $f(z_1,z_2,...,z_m,t)$ to belong to the domain of the generator $\mathcal{A}$, it is sufficient that $f(z_1,...,z_m,t)$ be twice differentiable with respect to z_1,...,z_m, t, for all z_1,...,z_m, t, and that

$$\left|\int_0^\infty f(z_1,...,z_i-x_i,...,z_m,t)dF_i(x_i)-f(z_1,...,z_m,t)\right|<\infty,\ \ i=1,...,m,$$

$$\left|\int_{[0,\infty)^2} f(z_1,...,z_i-x_i,...,z_j-x_j,...,z_m,t)dF_{i,j}(x_i,x_j)-f(z_1,...,z_m,t)\right|<\infty,$$

where $1\le i<j\le m$, and

$$\left|\int_{[0,\infty)^m} f(z_1-x_1,...,z_m-x_m,t)dF(x_1,x_2,...,x_m)-f(z_1,...,z_m,t)\right|<\infty.$$

$\square$

The following theorem provides the construction of a martingale using the result established in Proposition 4.7.1. Recall from Section 4.3 that according to Lemma 4.3.1, the set M is non-empty, where

$M=\{(r_1,..,r_m)\in[0,r_1^0)\times...\times[0,r_m^0)\mid M_{X_1,..,X_m}(r_1,..,r_m)<\infty\}-\{(0,..,0)\}.$

Theorem 4.7.1. *If $(r_1,...,r_m)\in M$, then the process*

$$W(t)=e^{-tg_1(r_1,...,r_m)}e^{-r_1V_1(t)-...-r_mV_m(t)},\quad t\ge 0$$

is a martingale with respect to the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},P)$, where $g_1(r_1,...,r_m)$ is defined as follows

$$g_1(r_1,...,r_m)=-\sum_{i=1}^{m}c_ir_i+\frac{1}{2}\sum_{i=1}^{m}\sigma_i^2r_i^2+\sum_{1\le i<j\le m}\sigma_i\sigma_j\rho_{ij}r_ir_j$$

$$+\sum_{i=1}^{m}\lambda_{ii}\left[M_{X_i}(r_i)-1\right]$$

$$+ \sum_{1 \le i < j \le m} \lambda_{ij} \left[M_{X_i,X_j}(r_i, r_j) - 1 \right] + \lambda_{1\ldots m} \left[M_{X_1,\ldots,X_m}(r_1, \ldots, r_m) - 1 \right]. \qquad (4.7.5)$$

PROOF. By Proposition 2.1.5, we have that for a function f belonging to the domain of the infinitesimal generator obtained in Proposition 4.7.1 such that $\mathcal{A}f = 0$, the process $\{f(V_1(t), \ldots, V_m(t), t), t \ge 0\}$ is a martingale. As in the proof of Theorem 4.3.1, we try a solution of the form $f(z_1, \ldots, z_m, t) = \beta(t) e^{-r_1 z_1 - \ldots - r_m z_m}$, where we can assume that $\beta(0) = 1$. Then the equation $\mathcal{A}f = 0$ yields

$$\beta'(t) + \beta(t) g_1(r_1, \ldots, r_m) = 0,$$

where $g_1(r_1, \ldots, r_m)$ is expressed as (4.7.5), and the solution is

$$\beta(t) = e^{-t g_1(r_1, \ldots, r_m)}.$$

Thus, the process $W(t) = e^{-t g_1(r_1, \ldots, r_m)} e^{-r_1 V_1(t) - \ldots - r_m V_m(t)}$ is a martingale. □

Remark 4.7.1. *Note that*

$$g_1(r_1, \ldots, r_m) = g(r_1, \ldots, r_m) + \frac{1}{2} \sum_{i=1}^{m} \sigma_i^2 r_i^2 + \sum_{1 \le i < j \le m} \sigma_i \sigma_j \rho_{ij} r_i r_j,$$

where $g(r_1, \ldots, r_m)$ was defined in Theorem 4.3.1 as relation (4.3.15).

With the aid of the exponential martingale obtained in Theorem 4.7.1, an upper bound for the ruin probability $\psi_{sim}(u_1, \ldots, u_m)$ is derived as follows.

Theorem 4.7.2. *Consider the risk model (4.7.1) and $g_1(r_1, \ldots, r_m)$ defined by relation (4.7.5) such that $\sup_{(r_1,\ldots,r_m) \in M} g_1(r_1, \ldots, r_m) > 0$.*

(1) If $\rho_{ij} \in [0, 1]$, except for at most one element, for all $i, j = 1, \ldots, m$, $i \ne j$, then

$$\psi_{sim}(u_1, \ldots, u_m) \le \inf_{(r_1,\ldots,r_m) \in S_1} e^{-\sum_{i=1}^{m} r_i u_i}, \qquad (4.7.6)$$

where $S_1 = \{(r_1, \ldots, r_m) \in M \mid g_1(r_1, \ldots, r_m) = 0\}$.

(2) If $\rho_{ij} \in [-1, 1]$ for all $i, j = 1, \ldots, m$, $i \ne j$, then

$$\psi_{sim}(u_1, \ldots, u_m) \le \inf_{(r_1,\ldots,r_m) \in S_2} e^{-\sum_{i=1}^{m} r_i u_i}, \qquad (4.7.7)$$

where $S_2 = \{(r_1, \ldots, r_m) \in M \mid g_1(r_1, \ldots, r_m) \le 0\}$.

Proof. We use similar arguments to those made in the proof of Theorem 4.4.1. Therefore, for a fixed time $t > 0$, the stopping time $\tau_{sim} \wedge t = \min(\tau_{sim}, t)$ is bounded by t and the martingale stopping property, given by Proposition 2.1.4, can be applied for the martingale $W(t)$ obtained in Theorem 4.7.1. Thus, for an arbitrary $(r_1, ..., r_m) \in M$, we have

$$e^{-r_1u_1-...-r_mu_m} = E[W(0)] = E[e^{-r_1V_1(\tau_{sim}\wedge t)-...-r_mV_m(\tau_{sim}\wedge t)-g_1(r_1,...,r_m)(\tau_{sim}\wedge t)}]$$

$$= E[e^{-r_1V_1(\tau_{sim})-...-r_mV_m(\tau_{sim})-g_1(r_1,...,r_m)\tau_{sim}}; \tau_{sim} \le t]$$

$$+E[e^{-r_1V_1(t)-...-r_mV_m(t)-g_1(r_1,...,r_m)t}; \tau_{sim} > t]$$

$$\ge E[e^{-r_1V_1(\tau_{sim})-...-r_mV_m(\tau_{sim})-g_1(r_1,...,r_m)\tau_{sim}}|\tau_{sim} \le t]\ P(\tau_{sim} \le t)$$

$$\ge E[e^{-g_1(r_1,...,r_m)\tau_{sim}}|\tau_{sim} \le t]\ P(\tau_{sim} \le t), \tag{4.7.8}$$

since $V_i(\tau_{sim}) < 0$ on $(\tau_{sim} < \infty)$, for all $i = 1, 2, ..., m$.

Letting $t \to \infty$ in (4.7.8) yields

$$\psi_{sim}(u_1, ..., u_m) \le e^{-r_1u_1-...-r_mu_m} \sup_{l\ge 0} e^{lg_1(r_1,...,r_m)},$$

and further, under the restriction $\sup_{l\ge 0} e^{lg_1(r_1,...,r_m)} < \infty$, it follows that

$$\psi_{sim}(u_1, ..., u_m) \le \inf_{(r_1,...,r_m)\in S_2} e^{-r_1u_1-...-r_mu_m}. \tag{4.7.9}$$

In order to complete the proof, we need to establish that the sets S_1 and S_2 are non-empty. We start by noting that $g_1(0, ..., 0) = 0$ and we want to examine the sign of $g_1(r_1, ..., r_m)$ around the origin. For this, consider that k_2,...,k_m are given non-negative real numbers, and let us denote

$$h_1(r_1) = g_1(r_1, k_2r_1, ..., k_mr_1).$$

By Remark 4.7.1, we have that

$$h_1(r_1) = g(r_1, k_2r_1, ..., k_mr_1) + \tfrac{1}{2}\sigma_1^2r_1^2 + \tfrac{1}{2}\sum_{i=2}^{m}\sigma_i^2k_i^2r_1^2$$

$$+\sum_{i=2}^{m}\sigma_1\sigma_i\rho_{1i}k_ir_1^2 + \sum_{2\le i<j\le m}\sigma_i\sigma_j\rho_{ij}k_ik_jr_1^2.$$

Using the results obtained within the proof of Theorem 4.3.2, the following properties regarding the first two derivatives of $h_1(r_1)$ are obtained as follows

$$\frac{dh_1(r_1)}{dr_1}\mid_{r_1=0}=\frac{dg(r_1,k_2r_1,...,k_mr_1)}{dr_1}\mid_{r_1=0}$$

$$+\left[\sigma_1^2 r_1+r_1\sum_{i=2}^{m}\sigma_i^2k_i^2+2r_1\sum_{i=2}^{m}\sigma_1\sigma_i\rho_{1i}k_i+2r_1\sum_{2\leq i<j\leq m}\sigma_i\sigma_j\rho_{ij}k_ik_j\right]\mid_{r_1=0}$$

$$=\left[(\sum_{j=1}^{m}\lambda_{1j}+\lambda_{1...m})\mu_1-c_1\right]+\sum_{i=2}^{m}\left[(\sum_{j=1}^{m}\lambda_{ij}+\lambda_{1...m})\mu_i-c_i\right]k_i<0,$$

in view of the net profit conditions (4.2.2), and

$$\frac{d^2h_1(r_1)}{dr_1^2}=\frac{d^2g(r_1,k_2r_1,...,k_mr_1)}{dr_1^2}$$

$$+\sigma_1^2+\sum_{i=2}^{m}\sigma_i^2k_i^2+2\sum_{i=2}^{m}\sigma_1\sigma_i\rho_{1i}k_i+2\sum_{2\leq i<j\leq m}\sigma_i\sigma_j\rho_{ij}k_ik_j, \tag{4.7.10}$$

where $\frac{d^2g(r_1,k_2r_1,...,k_mr_1)}{dr_1^2}>0$ (following the proof of Theorem 4.3.2).

Now, we consider two cases:

Case (1): If $\rho_{ij}\in[0,1]$, for all $i,j=1,...,m$, then from (4.7.10) it follows that $\frac{d^2h_1(r_1)}{dr_1^2}>0$, and the convexity property of the function h_1 is established.

If exactly only one element, say ρ_{ij}, is in $[-1,1]$, and all of the others are in $[0,1]$, then

$$\sigma_i^2k_i^2+2\sigma_i\sigma_j\rho_{ij}k_ik_j+\sigma_j^2k_j^2\geq(\sigma_ik_i-\sigma_jk_j)^2\geq 0.$$

Therefore, $\frac{d^2h_1(r_1)}{dr_1^2}>0$, which implies again that the function h_1 is convex.

Consequently, since k_2,...,k_m may be any non-negative constants, we obtain that along every ray from the origin into $[0,\infty)^m$, $g_1(r_1,r_2,...,r_m)$ is a continuous, decreasing function at zero, it is convex and such that $g_1(0,...,0)=0$. Therefore, $g_1(r_1,...,r_m)<0$ for all $(r_1,...,r_m)\in M$ from an arbitrary neighborhood of $(0,...,0)$, which together with continuity and the hypothesis of this proposition leads to the existence of at least one solution in M of the equation $g_1(r_1,r_2,...,r_m)=0$.

We thus conclude that the sets S_1 and S_2 are non-empty.

Moreover, for given k_2,...,k_m non-negative constants, the convexity of h_1 implies that if $(s,k_2s,...,k_ms)$ is a solution of the equation $g_1(r_1,r_2,...,r_m)=0$, then for $r_1<s$, $g_1(r_1,k_2r_1,...,k_mr_1)<0$ and for $r_1>s$, $g_1(r_1,k_2r_1,...,k_mr_1)>0$.

So, the infimum in (4.7.9) can be attained on S_1 which completes the proof of inequality (4.7.6).

Case (2): If $\rho_{ij} \in [-1, 1]$, for all $i, j = 1, ..., m$, $i \neq j$, then along every ray from the origin into $[0, \infty)^m$, $g_1(r_1, r_2, ..., r_m)$ is a continuous, decreasing function at zero and such that $g_1(0, ..., 0) = 0$. By similar arguments as in Case (1), we conclude that the equation $g_1(r_1, r_2, ..., r_m) = 0$ admits at least one solution in M and hence, the sets S_1 and S_2 are non-empty, but the infimum in (4.7.9) can be attained on S_1 or on S_2 as well. In this case, the inequality (4.7.7) is proved. $\square$

Remark 4.7.2. *Following the proof of Theorem* 4.7.2*, since for* $\rho_{ij} \in [0, 1]$*, except for at most one element, for all* $i, j = 1, ..., m$*,* $i \neq j$*, the function* h_1 *is convex, it results that for given* k_2*,...,*k_m *non-negative constants, the equation*

$$g_1(r_1, k_2 r_1, ..., k_m r_1) = 0$$

has a unique solution in $(0, r_1^0)$ *if* $\sup_{(r_1, k_2 r_1, ..., k_m r_1) \in M} g_1(r_1, k_2 r_1, .., k_m r_1) > 0$*.*

Remark 4.7.3. *Similar to Remark* 4.3.2*, if* $\rho_{ij} \in [0, 1]$*, except for at most one element, for all* $i, j = 1, ..., m$*,* $i \neq j$*,* $\lambda_{ii} \neq 0$*, for* $i = 1, ..., m$*, and* $M = [0, r_1^0) \times ... \times [0, r_m^0)$*, the condition*

$$\sup_{(r_1, ..., r_m) \in M} g_1(r_1, ..., r_m) > 0$$

imposed in Theorem 4.7.2 *is satisfied since again the following property holds:*

$$\lim_{r_1 \uparrow r_1^0, ..., r_m \uparrow r_m^0} g_1(r_1, ..., r_m) = \infty.$$

Indeed, by Remark 4.7.1*, we have that* $g_1(r_1, ..., r_m) \geq g(r_1, ..., r_m)$ *if* $\rho_{ij} \in [0, 1]$*, except for at most one element, for all* $i, j = 1, ..., m$*,* $i \neq j$*, and then Remark* 4.3.2 *is applied.*

In the univariate case ($m = 1$, $\rho_{11} = 1$), the equation $g_1(r_1) = 0$ becomes equation (2.3.33), namely,

$$-c_1 r_1 + \lambda_{11} \left(M_{X_1}(r_1) - 1 \right) + \tfrac{1}{2} \sigma_1^2 r_1^2 = 0.$$

As already mentioned in Subsection 2.3.5, the positive solution of this equation is the adjustment coefficient R introduced by Dufresne and Gerber (1991) for the classical risk model perturbed by a Brownian motion and therefore, their upper bound given by (2.3.34) is obtained by setting $m = 1$ in the upper bound described by (4.7.6)

In the bivariate case ($m = 2$), the second derivative of function $h_1(r_1)$ from the proof of Theorem 4.7.2 is strictly positive when $\rho_{12} \in [-1, 1]$, since using (4.7.10) we obtain

$$\begin{aligned} \tfrac{d^2 h(r_1)}{dr_1^2} &= \tfrac{d^2 g(r_1, k_2 r_1)}{dr_1^2} + \sigma_1^2 + \sigma_2^2 k_2^2 + 2\sigma_1\sigma_2\rho_{12}k_2 \\ &\geq \tfrac{d^2 g(r_1, k_2 r_1)}{dr_1^2} + (\sigma_1 - \sigma_2 k_2)^2 > 0. \end{aligned}$$

Therefore, the result *(i)* from Theorem 4.7.2 holds in the bivariate case when $\rho_{12} \in [-1, 1]$ and further, assuming that $\lambda_{11} = \lambda_{22} = 0$ and $\lambda_{12} = \lambda$, the result established by Li, Liu and Tang (2007), formulated in Chapter 2 as Proposition 2.4.8, is recovered.

The martingale obtained in Theorem 4.7.1 has a similar structure as the martingale obtained in Theorem 4.3.1 for the risk model (4.2.1). Therefore, the result from Theorem 4.5.1 can be applied in a straightforward way to the risk model (4.7.1), in view of the net profit conditions (4.2.2). This is formulated as follows.

Theorem 4.7.3. *Consider the risk model* (4.7.1) *and* $g_1(r_1, ..., r_m)$ *defined by* (4.7.5) *such that* $\sup_{(r_1,...,r_m)\in M} g_1(r_1, ..., r_m) > 0$ *and* $\rho_{ij} \in [-1, 1]$ *for all* $i, j = 1, ..., m$, $i \neq j$.

If $(r_1, .., r_m) \in S_1$, *where* $S_1 = \{(r_1, ..., r_m) \in M \mid g_1(r_1, ..., r_m) = 0\}$, *then*

$$\psi_{or}(u_1, ..., u_m) = \frac{e^{-r_1 u_1 - ... - r_m u_m}}{E[e^{-r_1 V_1(\tau_{or}) - ... - r_m V_m(\tau_{or})} \mid \tau_{or} < \infty]}.$$

We conclude this section by discussing the impact of the diffusion term in (4.7.1) on the upper bound of the ruin probability $\psi_{sim}(u_1, ..., u_m)$, given by (4.7.6).

4.7.1. The impact of perturbation

As in Li, Liu and Tang (2007), we have the following properties related to the impact of the perturbation on the upper bound of the ruin probability of type ψ_{sim}. Recall that the functions $g(r_1, r_2, ..., r_m)$ and $g_1(r_1, r_2, ..., r_m)$ are defined by (4.3.15) and (4.7.5), respectively.

The first property is devoted to comparing the upper bound given by Theorem 4.4.1 to the bound given by Theorem 4.7.2, case (1).

Property 1. Adding a diffusion process to the multivariate risk model (4.2.1) *leads to an increase in the upper bound for the ruin probability* $\psi_{sim}(u_1, ..., u_m)$.

Indeed, assume that the multivariate risk model perturbed by diffusion is given by (4.7.1), provided that $\rho_{ij} \in [0,1]$, except for at most one element, for all $i, j = 1, ..., m$, $i \neq j$. By Remarks 4.3.1 and 4.7.2 respectively, let k_i, $i = 2, ..., m$, be non-negative constants such that $(r_1', k_2 r_1', ..., k_m r_1')$ is the unique solution of the equation $g(r_1, k_2 r_1, ..., k_m r_1) = 0$ and $(r_1'', k_2 r_1'', ..., k_m r_1'')$ is the unique solution of the equation $g_1(r_1, k_2 r_1, ..., k_m r_1) = 0$. By Remark 4.7.1, it follows that

$$g_1(r_1', k_2 r_1', ..., k_m r_1') = \frac{1}{2}\sigma_1^2 (r_1')^2 + \frac{1}{2}\sum_{i=2}^{m} \sigma_i^2 k_i^2 (r_1')^2$$

$$+ \sum_{i=2}^{m} \sigma_1 \sigma_i \rho_{1i} k_i (r_1')^2 + \sum_{2 \leq i < j \leq m} \sigma_i \sigma_j \rho_{ij} k_i k_j (r_1')^2 \geq 0 = g_1(r_1'', k_2 r_1'', ..., k_m r_1''),$$

which further implies that $r_1' \geq r_1''$, since as in the proof of Theorem 4.7.2, $g_1(r_1, k_2 r_1, ..., k_m r_1) < 0$ for all $0 < r_1 < r_1''$ and $g_1(r_1, k_2 r_1, ..., k_m r_1) > 0$ for all $r_1 > r_1''$. Therefore, the upper bound given by Theorem 4.7.2, case (1), is greater than the one given by Theorem 4.4.1 , that is,

$$\inf e^{-r_1' u_1 - k_2 r_1' u_2 - ... - k_m r_1' u_m} \leq \inf e^{-r_1'' u_1 - k_2 r_1'' u_2 - ... - k_m r_1'' u_m},$$

where the infimum is taken over all $(k_2, ..., k_m) \in [0, \infty)^{m-1}$.

The following two properties discuss the impact of the correlation coefficients ρ_{ij} and of the volatility coefficients σ_i respectively, on the upper bound given by Theorem 4.7.2, case (1), associated to the risk model (4.7.1).

Property 2. The upper bound, from (4.7.6), *for the ruin probability* ψ_{sim} *of the multivariate risk model perturbed by diffusion* (4.7.1) *is increasing as the correlation coefficients* $(\rho_{ij})_{1 \leq i < j \leq m}$ *are increasing, under the assumption that* $\rho_{ij} \in [0,1]$ *except for at most one element, for all* $i, j = 1, ..., m$, $i \neq j$.

For this, we start by pointing out that the function $g_1(r_1, ..., r_m)$ is increasing in ρ_{ij} for all $(r_1, ..., r_m) \in M$ and for all $1 \leq i < j \leq m$, since

$$\frac{\partial g_1(r_1, ..., r_m)}{\partial \rho_{ij}} = \sum_{1 \leq i < j \leq m} \sigma_i \sigma_j r_i r_j \geq 0.$$

Hence, for fixed (i,j), $1 \le i < j \le m$, let $\rho_{ij}^{(1)} \le \rho_{ij}^{(2)}$. Also, for these values, the corresponding solutions of the equation $g_1(r_1, k_2 r_1, ..., k_m r_1) = 0$ for given $k_2, ..., k_m \ge 0$ are denoted by $(r_1^{(1)}, k_2 r_1^{(1)}, ..., k_m r_1^{(1)})$ and $(r_1^{(2)}, k_2 r_1^{(2)}, ..., k_m r_1^{(2)})$, respectively. Thus, the monotony of the function g_1 with respect to ρ_{ij} implies

$$g_1\left(r_1^{(1)}, k_2 r_1^{(1)}, ..., k_m r_1^{(1)}\right)|_{\rho_{ij}=\rho_{ij}^{(2)}} \ge g_1\left(r_1^{(1)}, k_2 r_1^{(1)}, ..., k_m r_1^{(1)}\right)|_{\rho_{ij}=\rho_{ij}^{(1)}} = 0$$
$$= g_1\left(r_1^{(2)}, k_2 r_1^{(2)}, ..., k_m r_1^{(2)}\right)|_{\rho_{ij}=\rho_{ij}^{(2)}}.$$

From this relation and the fact that $g_1(r_1, k_2 r_1, ..., k_m r_1)|_{\rho_{ij}=\rho_{ij}^{(2)}} < 0$ for all $0 < r_1 < r_1^{(2)}$ and $g_1(r_1, k_2 r_1, ..., k_m r_1)|_{\rho_{ij}=\rho_{ij}^{(2)}} > 0$ for all $r_1 > r_1^{(2)}$, it follows that $r_1^{(1)} \ge r_1^{(2)}$. Therefore,

$$\inf e^{-r_1^{(1)}u_1 - k_2 r_1^{(1)} u_2 - ... - k_m r_1^{(1)} u_m} \le \inf e^{-r_1^{(2)}u_1 - k_2 r_1^{(2)} u_2 - ... - k_m r_1^{(2)} u_m},$$

where the infimum is taken over all $(k_2, ..., k_m) \in [0, \infty)^{m-1}$.

Property 3. The upper bound, from (4.7.6)*, for the ruin probability ψ_{sim} of the multivariate risk model perturbed by diffusion* (4.7.1) *is increasing as the volatility coefficients $(\sigma_i)_{1\le i\le m}$ are increasing, under the assumption that $\rho_{ij} \in [0,1]$ for all $i, j = 1, ..., m$, $i \ne j$.*

By using the hypothesis that $\rho_{ij} \in [0,1]$ for all $1 \le i < j \le m$, it follows that the function $g_1(r_1, ..., r_m)$ is increasing in σ_i for all $(r_1, ..., r_m) \in M$ and for all $1 \le i \le m$, since

$$\frac{\partial g_1(r_1,...,r_m)}{\partial \sigma_i} = \sigma_i r_i^2 + \sum_{j\ne i} \sigma_j \rho_{ij} r_i r_j \ge 0.$$

From this point, the proof is identical to that of the Property 2 with ρ_{ij} replaced by σ_i.

4.8. Numerical illustrations for the trivariate case

Following Remarks 4.3.1 and 4.7.2, we consider numerical illustrations of the $(r_1, k_2 r_1, ..., k_m r_1)$-values with various non-negative k_2,...,k_m, and of the corresponding upper bounds given by Theorem 4.4.1 and by Theorem 4.7.2, case (1), for the particular case where $m = 3$. All the calculations were carried out with the software MATHEMATICA.

In order to better cover the dependence between the frequency of claims, we consider the trivariate risk model obtained from (4.2.1) by setting $m = 3$, which incorporates common shocks that affect couples of classes and also, all classes of business. Therefore, this model has the following structure

$$U_i(t) = u_i + c_i t - \sum_{k=1}^{N_i(t)} X_{ik}, \quad t \geq 0, \quad i = 1, 2, 3,$$

where the initial surplus and premium rate are denoted by u_i and c_i, respectively. The claims number processes $\{N_i(t), t \geq 0\}$, $i = 1, 2, 3$, follow a Poisson model with common shocks:

$$\begin{aligned} N_1(t) = &\quad N_{11}(t) + N_{12}(t) + N_{13}(t) + N_{123}(t), \\ N_2(t) = &\quad N_{22}(t) + N_{12}(t) + N_{23}(t) + N_{123}(t), \\ N_3(t) = &\quad N_{33}(t) + N_{13}(t) + N_{23}(t) + N_{123}(t), \end{aligned}$$

where $\{N_{ij}(t), t \geq 0\}$, for $1 \leq i \leq j \leq 3$, and $\{N_{123}(t), t \geq 0\}$ are all mutually independent Poisson processes with parameters λ_{ij} and λ_{123}, respectively.

The dependence between the claim sizes X_1, X_2, and X_3 is modeled using the notion of copula.

The copula approach to dependence modeling is rooted in a representation theorem due to Sklar (1959), stated as Theorem 2.4.1 in Subsection 2.4.1. Copulas have been known and studied for more than 45 years and only until relatively recently have been appearing in the actuarial literature. For example, papers by Frees and Valdez (1998), Wang (1998) and Cossette et al. (2008, 2010) discuss possible applications of copula theory to actuarial work.

In this study, we employ the trivariate Farlie-Gumbel-Morgenstern (FGM) family of copulas which allows only pairwise correlations, obtained by setting $n = 3$ in the multivariate FGM copula illustrated by Example 2.4.3 in Subsection 2.4.1. This family is popular in the literature and is attractive because of its simplicity and the fact that it allows both negative and positive dependence. For example, in house insurance where storm-water damage is covered, a storm in a single area would cause similar damages to the properties in that area, consequently generating claims of similar amounts.

Therefore, using Example 2.4.3, the trivariate FGM-copula function allowing only pairwise correlations is defined as

$$C(z_1, z_2, z_3) = \prod_{i=1}^{3} z_i \left[1 + \sum_{1\le i<j\le 3} \theta_{ij}(1-z_i)(1-z_j)\right], \; z_1, z_2, z_3 \in [0,1]. \quad (4.8.1)$$

Each copula of this family is absolutely continuous and has density

$$c(z_1, z_2, z_3) = 1 + \sum_{1\le i<j\le 3} \theta_{ij}(1-2z_i)(1-2z_j),$$

according to Definition 2.4.3. For the nonnegativity of the density function, θ_{ij} should satisfy the restriction

$$1 + \sum_{1\le i<j\le 3} \theta_{ij}(1-2z_i)(1-2z_j) \ge 0. \quad (4.8.2)$$

Thus, a sufficient condition is that $\left|\sum_{1\le i<j\le 3} \theta_{ij}\right| \le 1$.

In the general case of FGM n-copula defined by Example 2.4.3, Komelj and Perman (2010) showed that if all θ_{ij} lie in the interval $[-\frac{2}{n(n-1)}, \frac{2}{n(n-1)}]$, then the condition of type (4.8.2), namely:

$$1 + \sum_{1\le i<j\le n} \theta_{ij}(1-2z_i)(1-2z_j) \ge 0$$

is satisfied. Therefore, we assume that for $n = 3$, $\theta_{ij} \in [-\frac{1}{3}, \frac{1}{3}]$.

Assume also that X_i, $i = 1, 2, 3$, are exponential random variables with the distribution functions $F_i(x_i) = 1 - e^{-\alpha_i x_i}$ ($\alpha_i > 0$, $x_i > 0$). Then, in view of Theorem 2.4.1 (Sklar's theorem), the joint distribution function $F(x_1, x_2, x_3)$ of the vector (X_1, X_2, X_3) described by the trivariate FGM-copula has the form

$$F(x_1, x_2, x_3) = C(F_1(x_1), F_2(x_2), F_3(x_3)),$$

where $C(z_1, z_2, z_3)$ is given by (4.8.1). Consequently,

$$F(x_1, x_2, x_3) = \prod_{i=1}^{3}(1 - e^{-\alpha_i x_i}) \left[1 + \sum_{1\le i<j\le 3} \theta_{ij} e^{-\alpha_i x_i} e^{-\alpha_j x_j}\right],$$

and the joint density of (X_1, X_2, X_3) is

$$f(x_1, x_2, x_3) = \alpha_1\alpha_2\alpha_3 e^{-\alpha_1 x_1} e^{-\alpha_2 x_2} e^{-\alpha_3 x_3} \left[1 + \sum_{1\le i<j\le 3} \theta_{ij}(1-2x_i)(1-2x_j)\right]. \quad (4.8.3)$$

According to Komelj and Perman (2010), the correlation of X_i and X_j is

$$\rho(X_i, X_j) = \frac{\theta_{ij}}{4} \in \left[-\frac{1}{12}, \frac{1}{12}\right]. \tag{4.8.4}$$

If the dependence parameters θ_{ij} equal zero, then the random variables X_1, X_2, X_3 are all mutually independent.

Even though the range of dependence of FGM copula is quite restricted, it is of convenient form and allows pairwise correlations.

By Remark 2.4.1, for each (i, j) with $1 \leq i < j \leq 3$, the joint distribution function of (X_i, X_j) is described by the bivariate FGM-copula, given by Example 2.4.2, and hence, is expressed as:

$$F(x_i, x_j) = (1 - e^{-\alpha_i x_i})(1 - e^{-\alpha_j x_j})\left(1 + \theta_{ij} e^{-\alpha_i x_i} e^{-\alpha_j x_j}\right),$$

while the joint density of (X_1, X_2) has the form

$$f(x_i, x_j) = \alpha_1 \alpha_2 e^{-\alpha_1 x_1} e^{-\alpha_2 x_2} \left[1 + \theta_{ij}(1 - 2x_i)(1 - 2x_j)\right]. \tag{4.8.5}$$

In this setting, we want to compute solutions of the form $(r_1, k_2 r_1, k_3 r_1)$ for the equation $g(r_1, r_2, r_3) = 0$, obtained from (4.3.15) by letting $m = 3$, in order to obtain results for the upper bounds of the ruin probability $\psi_{sim}(u_1, u_2, u_3)$, as in Theorem 4.4.1. The equation $g(r_1, r_2, r_3) = 0$ is equivalent to

$$\begin{aligned} & -\sum_{i=1}^{3} c_i r_i + \sum_{i=1}^{3} \lambda_{ii} \left[M_{X_i}(r_i) - 1\right] \\ & + \sum_{1 \leq i < j \leq 3} \lambda_{ij} \left[M_{X_i, X_j}(r_i, r_j) - 1\right] + \lambda_{123}[M_{X_1, X_2, X_3}(r_1, r_2, r_3) - 1] = 0, \end{aligned} \tag{4.8.6}$$

and we need to evaluate the joint moment generating function of (X_1, X_2, X_3) and of (X_i, X_j). First, we notice that for each $i = 1, 2, 3$, the moment generating function of X_i is equal to

$$M_{X_i}(r_i) = E[e^{r_i X_i}] = \int_0^\infty \alpha_i e^{r_i x_i} e^{-\alpha_i x_i} dx = \frac{\alpha_i}{\alpha_i - r_i}, \quad r_i < \alpha_i. \tag{4.8.7}$$

Hence, $r_i^0 = \alpha_i$, $i = 1, 2, 3$. For $1 \leq i < j \leq 3$, using the density $f(x_1, x_2)$ from (4.8.5), we obtain

$$M_{X_i, X_j}(r_i, r_j) = E[e^{r_i X_i + r_j X_j}] = \int_0^\infty \int_0^\infty e^{r_i x_i} e^{r_j x_j} f(x_i, x_j) dx_i dx_j$$

$$= \alpha_i \alpha_j \frac{(2\alpha_i - r_i)(2\alpha_j - r_j) + \theta_{ij} r_i r_j}{(\alpha_i - r_i)(2\alpha_i - r_i)(\alpha_j - r_j)(2\alpha_j - r_j)}, \tag{4.8.8}$$

for $r_i < \alpha_i$, $i = 1, 2, 3$. Using the density $f(x_1, x_2, x_3)$ from (4.8.3) yields

$$M_{X_1,X_2,X_3}(r_1, r_2, r_3) = E[e^{r_1X_1 + r_2X_2 + r_3X_3}]$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty e^{r_1x_1} e^{r_2x_2} e^{r_3x_3} f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

$$= \frac{\alpha_3}{\alpha_3 - r_3} M_{X_1,X_2}(r_1, r_2) + \frac{\alpha_2}{\alpha_2 - r_2} M_{X_1,X_3}(r_1, r_3)$$

$$+ \frac{\alpha_1}{\alpha_1 - r_1} M_{X_2,X_3}(r_2, r_3) - \frac{2\alpha_1\alpha_2\alpha_3}{(\alpha_1 - r_1)(\alpha_2 - r_2)(\alpha_3 - r_3)}, \tag{4.8.9}$$

where the joint moment generating functions $M_{X_i,X_j}(r_i, r_j)$ for $1 \le i < j \le 3$ are given by (4.8.8). Thus, from (4.8.7), (4.8.8) and (4.8.9), it results that the sets M and M_{ij} from Lemma 4.3.1 are equal to

$$M = M_{ij} = [0, \alpha_1) \times [0, \alpha_2) \times [0, \alpha_3).$$

By using this result and Remark 4.3.2, we obtain that the condition $\sup_{(r_1,r_2,r_3) \in M} g(r_1, r_2, r_3) > 0$ from Theorem 4.4.1 is satisfied.

For example, for the three classes of business, we will assume that the mean claim sizes are $1/\alpha_1 = 5$, $1/\alpha_2 = 1$, $1/\alpha_3 = 10$, and the safety loading coefficients are $\theta_1 = 0.5$, $\theta_2 = 0.4$, $\theta_3 = 0.6$, respectively.

Four different cases that illustrate the arrival rates of the shocks are considered as follows:

Case 1. $\lambda_{11} = \lambda_{22} = \lambda_{33} = 11$; $\lambda_{12} = \lambda_{13} = \lambda_{23} = 0$; $\lambda_{123} = 0$;

Case 2. $\lambda_{11} = \lambda_{22} = \lambda_{33} = 5$; $\lambda_{12} = \lambda_{13} = \lambda_{23} = 3$; $\lambda_{123} = 0$;

Case 3. $\lambda_{11} = \lambda_{22} = \lambda_{33} = 5$; $\lambda_{12} = \lambda_{13} = \lambda_{23} = 2$; $\lambda_{123} = 2$;

Case 4. $\lambda_{11} = \lambda_{22} = \lambda_{33} = 5$; $\lambda_{12} = \lambda_{13} = \lambda_{23} = 0$; $\lambda_{123} = 6$.

Case 1 represents the independent model without any common shocks. Case 2 describes the model with common shocks that affect on each couple of classes of business, whereas Case 4 assumes only the existence of individual shocks and a common shocks that affect all three classes of business simultaneously. Case 3 deals with both the situations described by Cases 2 and 4. In Section 4.2, we exemplified these cases with possible situations in an insurance context.

The values of λ's have been chosen such that the total arrival rates for the three classes of business are the same: $\lambda_1 = \lambda_2 = \lambda_3 = 11$ and therefore, we want to see the impact of the common shocks on the upper bounds of the ruin probability ψ_{sim}. The premium rates, computed as $c_i = (1+\theta_i)\lambda_i/\alpha_i$, have the following values $c_1 = 82.5$, $c_2 = 15.4$, and $c_3 = 176$.

Since $E[\sum_{k=1}^{N_1(1)} X_{1k}] = \lambda_1 E[X_1] = 55$, $E[\sum_{k=1}^{N_2(1)} X_{2k}] = \lambda_2 E[X_2] = 11$ and $E[\sum_{k=1}^{N_3(1)} X_{3k}] = \lambda_3 E[X_3] = 110$, we set the following values for the initial surpluses of the three classes: $u_1 = 55$, $u_2 = 15$, and $u_3 = 120$, respectively.

Now, for these four cases, using formulas (4.8.7), (4.8.8) and (4.8.9), we compute solutions of the form (r_1, k_2r_1, k_3r_1) of equation (4.8.6) for various values of the slopes (k_2, k_3) such as $(0.1, 0.2)$, $(1,1)$, $(10,100)$, and $(0.1, 10)$, and of the coefficients θ_{ij} such as -0.2, 0, and 0.2. Note that, by relation (4.8.4), θ_{ij} is a measure of the interdependence of random variables X_i and X_j modeling the claim sizes, a positive value of θ_{ij} indicating positive correlation, a negative value indicating a negative correlation and absence of correlation at zero. For each value obtained for (r_1, k_2r_1, k_3r_1), the corresponding upper bound is computed as $e^{-r_1u_1-k_2r_1u_2-k_3r_1u_3}$.

Cases 1, 2, 3, and 4 are illustrated by Tables 4.1, 4.2, 4.3 and 4.4, respectively, where values of r_1 and of the corresponding upper bounds $e^{-r_1u_1-k_2r_1u_2-k_3r_1u_3}$ with $u_1 = 55$, $u_2 = 15$, and $u_3 = 120$ are presented.

We observe that as we move from Case i to Case $i+1$, $(i = 1, 2, 3)$, that is, from independence to dependence in loss frequencies across types, the upper bound of the ruin probability $\psi_{sim}(u_1, u_2, u_3)$ increases, regardless the correlation between the claims.

We prove now that the upper bound established in Theorem 4.4.1 increases as the correlation coefficients θ_{ij} increase.

Using relation (4.8.8), we note that for each $1 \le i < j \le 3$, $M_{X_i,X_j}(r_i, r_j)$ is increasing in θ_{ij} and hence, $M_{X_1,X_2,X_3}(r_1, r_2, r_3)$, given by (4.8.9), is increasing in θ_{ij}, which leads to the conclusion that $g(r_1, r_2, r_3)$ is increasing in θ_{ij}. For $1 \le i < j \le 3$, let $\theta_{ij}^{(1)} \le \theta_{ij}^{(2)}$ and for given k_2, k_3 non-negative constants,

TABLE 4.1. **Case 1**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$.

$k_2 = 0.1$ $k_3 = 0.2$	$k_2 = 1$ $k_3 = 1$	$k_2 = 10$ $k_3 = 100$	$k_2 = 0.1$ $k_3 = 10$
0.0803307	0.0428329	0.0003775	0.0038439
0.0015854	0.0002994	0.0101191	0.0081062

TABLE 4.2. **Case 2**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$.

$\theta_{12} = \theta_{13} = \theta_{23}$	$k_2 = 0.1$ $k_3 = 0.2$	$k_2 = 1$ $k_3 = 1$	$k_2 = 10$ $k_3 = 100$	$k_2 = 0.1$ $k_3 = 10$
-0.2	0.0750872 0.0024149	0.0390314 0.000615339	0.0003766 0.0102305	0.0038127 0.00842932
0	0.0747459 0.00248201	0.0387878 0.000644398	0.0003765 0.0102431	0.0038107 0.00845046
0.2	0.0744091 0.00255002	0.0385487 0.000674254	0.0003764 0.0102555	0.0038086 0.00847272

$(r_1^{(l)}, k_2r_1^{(l)}, k_3r_1^{(l)})$ be solutions of the equation $g(r_1, r_2, r_3)|_{\theta_{ij}=\theta_{ij}^{(l)}} = 0$, $l = 1, 2$. Then, for $1 \leq i < j \leq 3$, the monotony of g as a function of θ_{ij} yields

$$g\left(r_1^{(1)}, k_2r_1^{(1)}, ..., k_mr_1^{(1)}\right)|_{\theta_{ij}=\theta_{ij}^{(2)}} \geq g\left(r_1^{(1)}, k_2r_1^{(1)}, ..., k_mr_1^{(1)}\right)|_{\theta_{ij}=\theta_{ij}^{(1)}} = 0$$
$$= g\left(r_1^{(2)}, k_2r_1^{(2)}, ..., k_mr_1^{(2)}\right)|_{\theta_{ij}=\theta_{ij}^{(2)}}.$$

From this relation and the fact that $g(r_1, k_2r_1, ...k_mr_1)|_{\theta_{ij}=\theta_{ij}^{(2)}} < 0$ for all $0 < r_1 < r_1^{(2)}$ and $g(r_1, k_2r_1, ...k_mr_1)|_{\theta_{ij}=\theta_{ij}^{(2)}} > 0$ for all $r_1 > r_1^{(2)}$, it follows that $r_1^{(1)} \geq r_1^{(2)}$. Thus,

$$e^{-r_1^{(1)}u_1-k_2r_1^{(1)}u_2-k_3r_1^{(1)}u_3} \leq e^{-r_1^{(2)}u_1-k_2r_1^{(2)}u_2-k_3r_1^{(2)}u_3}.$$

This property is illustrated by the results given by each of the Tables 4.2 to 4.4. For example, in Table 4.2, for the choice of $(k_2, k_3) = (0.1, 0.2)$, as the coefficients θ_{ij} increase, the values of $e^{-r_1u_1-k_2r_1u_2-k_3r_1u_3}$ are 0.0024149, 0.00248201, and 0.00255002, respectively.

Table 4.3. **Case 3**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$.

$\theta_{12}=\theta_{13}=\theta_{23}$	$k_2=0.1$ $k_3=0.2$	$k_2=1$ $k_3=1$	$k_2=10$ $k_3=100$	$k_2=0.1$ $k_3=10$
-0.2	0.0734921 0.0027447	0.0378766 0.0007657	0.0003763 0.0102681	0.0038024 0.0085387
0	0.0730589 0.0028418	0.0375652 0.0008123	0.0003762 0.0102805	0.0037997 0.0085677
0.2	0.0726332 0.0029406	0.0372617 0.0008603	0.0003761 0.0102932	0.0037970 0.00859674

Table 4.4. **Case 4**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$.

$\theta_{12}=\theta_{13}=\theta_{23}$	$k_2=0.1$ $k_3=0.2$	$k_2=1$ $k_3=1$	$k_2=10$ $k_3=100$	$k_2=0.1$ $k_3=10$
-0.2	0.0705261 0.0034824	0.0357988 0.0011351	0.0003757 0.0103432	0.0037820 0.0087598
0	0.0699395 0.0036502	0.0353847 0.0012277	0.0003755 0.0103684	0.0037780 0.0088038
0.2	0.0693669 0.0038219	0.0349851 0.0013242	0.0003754 0.0103812	0.0037740 0.00884802

In order to illustrate the upper bound established in Theorem 4.7.2, case (1), we add a three-dimensional correlated Brownian motion to the trivariate risk model, assuming that the correlation coefficients $\rho_{ij} \in [0,1]$ for all $i,j=1,...,m$, $i \neq j$. By Remark 4.7.2, we aim to compute solutions of the form (r_1, k_2r_1, k_3r_1) for the equation $g_1(r_1, k_2r_1, k_3r_1)=0$, which is equivalent to

$$g(r_1,k_2r_1,k_3r_1)+\frac{r_1^2}{2}(\sigma_1^2+\sigma_2^2k_2^2+\sigma_3^2k_3^2)$$

$$+r_1^2(\sigma_1\sigma_2\rho_{12}k_2+\sigma_1\sigma_3\rho_{13}k_3+\sigma_2\sigma_3\rho_{23}k_2k_3)=0, \tag{4.8.10}$$

in view of Remark 4.7.1.

Again, since $M=[0,\alpha_1)\times[0,\alpha_2)\times[0,\alpha_3)$, by Remark 4.7.3, it results that the condition $\sup_{(r_1,r_2,r_3)\in M} g_1(r_1,r_2,r_3)>0$ from Theorem 4.7.2 is satisfied.

TABLE 4.5. **Case 3**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$ if $\theta_{12} = \theta_{13} = \theta_{23} = -0.2$.

$(\sigma_1, \sigma_2, \sigma_3)$ $(\rho_{12}, \rho_{13}, \rho_{23})$	$k_2 = 0.1$ $k_3 = 0.2$	$k_2 = 1$ $k_3 = 1$	$k_2 = 10$ $k_3 = 100$	$k_2 = 0.1$ $k_3 = 10$
$(0.1, 0.1, 0.1)$ $(0, 0, 0)$	0.07349171 0.00274485	0.03787641 0.000765824	0.00037629 0.01026924	0.00380239 0.00853889
$(0.1, 0.1, 0.1)$ $(0.1, 0.4, 0.6)$	0.07349162 0.00274487	0.03787630 0.00076584	0.00037628 0.01027052	0.00380236 0.00853921
$(0.3, 0.5, 0.8)$ $(0, 0, 0)$	0.07349091 0.00274502	0.03787211 0.00076644	0.00037626 0.01027335	0.00380207 0.00854231
$(0.3, 0.5, 0.8)$ $(0.1, 0.4, 0.6)$	0.07348882 0.00274548	0.0378688 0.00076692	0.00037625 0.01027426	0.00380195 0.00854362

Moreover, for illustrating Property 1 from Subsection 4.7.1, it is sufficient to consider only one of the above cases, for example Case 3, which contains all types of common shocks. In the context of Case 3, assuming all the other parameters remain unchanged, we consider various values of σ's and of ρ's. The results obtained for r_1, by solving equation (4.8.10), and of upper bounds $e^{-r_1u_1-k_2r_1u_2-k_3r_1u_3}$ ($u_1 = 55$, $u_2 = 15$, $u_3 = 120$) are given in Tables 4.3, 4.4 and 4.5 corresponding to the situations when $\theta_{12} = \theta_{13} = \theta_{23} = -0.2$, $\theta_{12} = \theta_{13} = \theta_{23} = 0$, and $\theta_{12} = \theta_{13} = \theta_{23} = 0.2$, respectively.

The results in Tables 4.5, 4.6 and 4.7 confirm that increasing the correlation coefficients $\rho_{ij} \in [0, 1]$, or increasing the volatility coefficients $\sigma_i \geq 0$ lead to increasing the upper bound of the ruin probability. This was established by Properties 2 and 3 from Subsection 4.7.1, respectively.

Also, increasing the dependence between claims yields an increase in the upper bound. Indeed, first we note that since $g(r_1, r_2, r_3)$ is increasing in θ_{ij}, then by Remark 4.7.1, it results that $g_1(r_1, r_2, r_3)$ is increasing in θ_{ij}. From this point, the proof is obtained from the proof of the same property regarding the trivariate risk model without diffusion, presented in this section, by replacing g with g_1.

TABLE 4.6. **Case 3**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$ if $\theta_{12} = \theta_{13} = \theta_{23} = 0$.

$(\sigma_1,\sigma_2,\sigma_3)$ $(\rho_{12},\rho_{13},\rho_{23})$	$k_2 = 0.1$ $k_3 = 0.2$	$k_2 = 1$ $k_3 = 1$	$k_2 = 10$ $k_3 = 100$	$k_2 = 0.1$ $k_3 = 10$
$(0.1, 0.1, 0.1)$ $(0, 0, 0)$	0.07305851 0.00284195	0.03756512 0.00081233	0.00037612 0.01029052	0.00379972 0.00856749
$(0.1, 0.1, 0.1)$ $(0.1, 0.4, 0.6)$	0.07305843 0.00284197	0.03756503 0.00081235	0.00037611 0.01029177	0.00379958 0.0085699
$(0.3, 0.5, 0.8)$ $(0, 0, 0)$	0.07305772 0.00284213	0.03756085 0.00081299	0.00037608 0.01029554	0.00379914 0.00857372
$(0.3, 0.5, 0.8)$ $(0.1, 0.4, 0.6)$	0.07305566 0.00284265	0.03755763 0.00081349	0.00037605 0.01029925	0.00379911 0.00857404

TABLE 4.7. **Case 3**: Values of r_1 and $e^{-55r_1-15k_2r_1-120k_3r_1}$ if $\theta_{12} = \theta_{13} = \theta_{23} = 0.2$

$(\sigma_1,\sigma_2,\sigma_3)$ $(\rho_{12},\rho_{13},\rho_{23})$	$k_2 = 0.1$ $k_3 = 0.2$	$k_2 = 1$ $k_3 = 1$	$k_2 = 10$ $k_3 = 100$	$k_2 = 0.1$ $k_3 = 10$
$(0.1, 0.1, 0.1)$ $(0, 0, 0)$	0.07263271 0.00294075	0.03372516 0.00086204	0.00037604 0.01030056	0.00379603 0.00860719
$(0.1, 0.1, 0.1)$ $(0.1, 0.4, 0.6)$	0.07263261 0.00294077	0.03725053 0.00086221	0.00037601 0.01030437	0.00379401 0.00862912
$(0.3, 0.5, 0.8)$ $(0, 0, 0)$	0.0726326 0.00294088	0.0372473 0.00086274	0.00037590 0.01031815	0.00379261 0.00864414
$(0.3, 0.5, 0.8)$ $(0.1, 0.4, 0.6)$	0.07262825 0.0029418	0.03724427 0.00086323	0.00037571 0.01034197	0.00379252 0.00864512

This property is illustrated by comparing Tables 4.5, 4.6 and 4.7 for each choice of (k_2, k_3), $(\sigma_1, \sigma_2, \sigma_3)$, and of $(\rho_{12}, \rho_{13}, \rho_{23})$.

Comparing Tables 4.1 to 4.7, we also note that the smallest values of the upper bounds were obtained when $(k_2 = 1, k_3 = 1)$, and the largest ones when $(k_2 = 10, k_3 = 100)$.

4.9. Conclusions

In this chapter, we investigated a multivariate risk process where the claims number processes follow a Poisson model with common shocks and dependence between claims is allowed.

With the aid of tools from piecewise deterministic Markov processes theory, we showed how to derive the exponential martingales needed to establish computable bounds on multivariate ruin probabilities. Based on these martingales, an upper bound for the probability that ruin occurs in all classes simultaneously is obtained and also, an expression for the probability that ruin occurs in at least one class of business is derived.

Assuming that the claim sizes are represented by dependent heavy-tailed random variables, we obtained an asymptotic upper bound for the probability that ruin occurs in all classes simultaneously, before a fixed time $t > 0$. For the case with independent heavy-tailed claims, an asymptotic result for this ruin probability is derived in a similar manner as in the univariate case.

The same approach using Markov processes theory was adopted for deriving exponential martingales in the case where a multidimensional Brownian process with a joint correlation matrix is added to this multivariate risk model. Similarly to the unperturbed multivariate risk model, we derived an upper bound for the probability that ruin occurs in all classes simultaneously, under the assumption that the correlation coefficients of the components from the multidimensional Brownian process are non-negative, except for at most one element.

We illustrated the results obtained on the upper bounds for the probability that ruin occurs in all classes simultaneously considering the trivariate case. For modeling the dependence between claim sizes we employed the Farlie-Gumbel-Morgenstern (FGM) family of copulas and considered that the claim sizes are exponentially distributed. We discussed the impact of the dependence parameters on the upper bounds and reached the conclusion that increasing the dependence parameters leads to increasing these bounds.

Chapter 5

RUIN PROBABILITIES IN A MULTIVARIATE RENEWAL MODEL

5.1. Introduction

Classical estimations for ruin probabilities are based on the fact that the number of claims up to time t constitutes a Poisson process, implying that the elapsed time since the last event does not influence the time of the next event. In reality, this assumption seems unlikely. For example, the atmosphere and the ocean can store extremal conditions over years and natural catastrophes may occur, like the outbreak of tornados that happened in North America in 2011 and produced damages to property, automobiles and interruption of businesses collaterally. Insurance companies are concerned about a trend in extreme weather, which may affect more than two types of insurance policies in a portfolio.

Therefore, the compound Poisson model does not always give a good description of reality and in this sense, there has been a significant volume of literature that questions the suitability of the Poisson process in insurance modeling; for example, Seal (1983), Beard et al. (1984).

Based on these facts, it is reasonable to think of a renewal process (possibly different from a Poisson process) to count the claims produced by a common shock, such as a natural disaster that affects all classes of insurance business.

Our contribution in this chapter is outlined by investigating ruin probabilities associated to an m-dimensional risk process which assumes that in addition to the independent underlying risks for each class of business, there are aggregate claims produced by a common shock that affects all classes of business. Assuming that the individual claim arrivals for each class of business are governed by Poisson processes, while the claim arrivals due to the common shock are governed by a common renewal process gives a more realistic model. It is also more challenging than the case studied in Chapter 4, where all common shocks are governed by Poisson processes.

In this multivariate setting, we derive upper bounds of Lundberg-type for the probability that ruin occurs in all classes simultaneously, denoted $\psi_{sim}(u_1, ..., u_m)$ and introduced by Definition 2.4.1.

In our study, the surplus vector process becomes a Markov process by introducing a supplementary process. Tools from the theory of piecewise deterministic Markov processes are applied in order to derive exponential martingales needed to establish upper bounds for the ruin probability of type $\psi_{sim}(u_1, ..., u_m)$.

The rest of the chapter consists of the following sections.

In Section 5.2, the multivariate model of study is introduced along with a special case where the individual shocks are absent and the claims across classes are generated only by a common renewal process. For the univariate versions of these two models, a review of the related results is presented in Subsection 5.2.1.

In Section 5.3, the backward Markovization technique and exponential martingales are provided, while in Section 5.4, we obtain upper bounds for the probability that ruin occurs in all classes simultaneously assuming the two models introduced in Section 5.2. Numerical results are presented in Section 5.5 for a special bivariate case based only on common shocks and where the dependence structure between the claim sizes is modeled by the bivariate Farlie-Gumbel-Morgenstern (FGM) copula. Section 5.6 concludes the chapter.

5.2. Multivariate renewal model formulation

Assume that a portfolio with m ($m \geq 1$) possibly dependent classes of business is modeled by a risk process characterized by the vector $\mathbf{U}(t) = (U_1(t), ..., U_m(t))$ of surplus processes with

- initial capital vector $\mathbf{u} = (u_1, ..., u_m)$, $u_i \geq 0$,

• a vector of m independent Poisson processes $\mathbf{M}(t) = (N_1(t), ..., N_m(t))$, such that for each $i = 1, ..., m$, $N_i(t)$ governs the occurrence of the claims corresponding to class i with severities $\{Y_{ik}\}_{k\geq 1}$,

• one ordinary renewal process $N(t)$ that generates a claim in each of the components represented through the claim vector $\mathbf{X}_k = (X_{1k}, ..., X_{mk})$, $k \geq 1$,

• and constant premium rate vector $\mathbf{c} = (c_1, ..., c_m)$, $c_i > 0$.

Therefore, this m-dimensional risk process is described by

$$\mathbf{U}(t) = \mathbf{u} + \mathbf{c}t - \sum_{k=1}^{\mathbf{M}(t)} \mathbf{Y}_k - \sum_{k=1}^{N(t)} \mathbf{X}_k, \quad t \geq 0, \tag{5.2.1}$$

where

$$\sum_{k=1}^{\mathbf{M}(t)} \mathbf{Y}_k = \left(\sum_{k=1}^{N_1(t)} Y_{1k}, ..., \sum_{k=1}^{N_m(t)} Y_{mk} \right) \text{and} \sum_{k=1}^{N(t)} \mathbf{X}_k = \left(\sum_{k=1}^{N(t)} X_{1k}, ..., \sum_{k=1}^{N(t)} X_{mk} \right).$$

We also assume that $\{N(t), t \geq 0\}$, $\{N_1(t), t \geq 0\}$,...,$\{N_m(t), t \geq 0\}$, $\{Y_{1k}\}_{k\geq 1}$, ...,$\{Y_{mk}\}_{k\geq 1}$ and $\{(X_{1k}, ..., X_{mk})\}_{k\geq 1}$ are all mutually independent.

In this model, in addition to the independent underlying risks $\{Y_{ik}\}_{k\geq 1}$ generated by the Poisson processes $\{N_i(t), t \geq 0\}$, for $1 \leq i \leq m$, there are claims generated by a common renewal process $\{N(t), t \geq 0\}$. A natural interpretation is that $N(t)$ represents the number of claims due to common shocks that affect all m classes of business, causing claims of sizes X_{ik}, in the i-th class. This allows for dependence of the random variables X_{1k},...,X_{mk}.

A typical example is that a severe car accident may cause not only the loss of the damaged car but also medical expenses of the injured driver and passengers. Also, in the case of a catastrophe such as an earthquake or a strong windstorm for example, the damages covered by homeowners and private passenger automobile insurance may be dependent.

Note that one might be interested in modeling a subportfolio with m possibly dependent classes of business based only on common shocks. For example, a common event such as a natural disaster may cause various kinds of insurance claims. Therefore, in this scenario, the risk process is obtained from (5.2.1) by letting $N_i(t) \equiv 0$ for $i = 1, ..., m$, and is characterized by

$$\mathbf{U}(t) = \mathbf{u} + \mathbf{c}t - \sum_{k=1}^{N(t)} \mathbf{X}_k, \quad t \geq 0. \tag{5.2.2}$$

For this reason, each result we derive for the risk model (5.2.1) will be followed by the corresponding result for the risk model defined by (5.2.2).

In what follows, we give a description of the stochastic quantities involved in defining the risk model (5.2.1).

For each $i = 1, ..., m$, $\{N_i(t), t \geq 0\}$ is assumed to be an homogeneous Poisson process with intensity λ_i.

Following the notation introduced in Section 2.2, for the ordinary renewal process $N(t)$ we have that $\{\sigma_n\}_{n\geq 1}$ denote the claim arrival times, where $0 = \sigma_0 < \sigma_1 < \sigma_2 < ...$ and $\{T_n\}_{n\geq 1}$ denote the claim inter-arrival times, where $T_1 = \sigma_1$ and $T_n = \sigma_n - \sigma_{n-1}$, $n = 2, 3, ...$

Since $N(t)$ is an ordinary renewal process, $\{T_n\}_{n\geq 1}$ are positive independent and identically distributed random variables and we will assume that their common distribution function is $Q(x)$ satisfying $\overline{Q}(x) = 1 - Q(x) > 0$ and common mean is $E[T_n] = 1/\lambda > 0$. It is further assumed that the distribution $Q(x)$ is absolutely continuous with the probability density function $q(x) = Q'(x)$. This assumption leads to defining the hazard rate function of the random times T_n as

$$\lambda(x) = \frac{q(x)}{1 - Q(x)} \ . \tag{5.2.3}$$

Recall from Subsection 2.2.1.2, that the expected number of renewals up to time t is represented by $m(t) = E[N(t)]$, which is finite for all $t \geq 0$. According to Proposition 2.2.7, we have that

$$\frac{m(t)}{t} \to \lambda \ \text{ as } \ t \to \infty. \tag{5.2.4}$$

For each $i = 1, .., m$, the claims generated by the Poisson process $N_i(t)$ are assumed to be modeled by a sequence of independent and identically distributed random variables $\{Y_{ik}\}_{k\geq 1}$ with common distribution function $G_i(x)$ such that $G_i(0) = 0$ and with common finite mean ν_i.

The claims generated by the renewal process $N(t)$ are assumed to be modeled by a sequence of independent and identically distributed m-dimensional positive random vectors $\{\mathbf{X}_k\}_{k\geq 1} = \{(X_{1k}, ..., X_{mk})\}_{k\geq 1}$ with common continuous joint distribution function $F(x_1, ..., x_m)$ and marginal distribution functions $F_i(x_i)$, $i = 1, ..., m$, satisfying $F_i(0) = 0$ and with common finite mean μ_i.

Throughout this chapter, for simplicity, let T represent an arbitrary T_k, Y_i represent an arbitrary Y_{ik}, $i = 1, ..., m$, and $\mathbf{X} = (X_1, ..., X_m)$ represent an arbitrary $\mathbf{X}_k = (X_{1k}, ..., X_{mk})$. We also assume that if $\mathbf{a} = (a_1, ..., a_m)$ and $\mathbf{b} = (b_1, ..., b_m)$ are two vectors from $\mathbb{R}^m$, then

i. $\mathbf{a} = \mathbf{b}$ if and only if $a_i = b_i$ for $i = 1, ..., m$, and

ii. $\mathbf{a} < \mathbf{b}$ if and only if $a_i < b_i$ for $i = 1, ..., m$.

According to Definitions 2.2.2 and 2.2.4 from Section 2.2, the time of ruin for the i-th class, for $i = 1, ..., m$, is defined as

$$\tau_i = \inf\{t \geq 0 : U_i(t) < 0\},$$

and the corresponding ruin probability as

$$\psi_i(u_i) = P(\tau_i < \infty \mid U_i(0) = u_i).$$

If for each $i = 1, ..., m$, the surplus $U_i(t) \geq 0$ for all $t \geq 0$ (no ruin occurs), we indicate this by writing $\tau_i = \infty$. To avoid the certainty of ruin, we assume that the net profit condition (2.2.4) is satisfied for each class of business, which in the context of the model given by (5.2.1) becomes

$$c_i E[T] - \lambda_i E[Y_i] E[T] - E[X_i] > 0, \text{ or equivalently, } c_i - \lambda_i \nu_i - \lambda \mu_i > 0, \quad (5.2.5)$$

for $i = 1, ..., m$. Relation (5.2.5) is easily established since $\lim_{t\to\infty} E[U(t)]/t = c_i - \lambda_i E[Y_i] - E[X_i] \lim_{t\to\infty}\{m(t)/t\}$ and property (5.2.4) can be applied.

Different ruin concepts in a multivariate setting were introduced via Definition 2.4.1 from Section 2.4. As it was illustrated in Subsection 2.4.2 and in Section 4.2, the problem involving the ruin probability of type ψ_{sum} can be reduced to a one-dimensional ruin problem.

Under the multivariate setting given by (5.2.1), the ruin probability ψ_{sum} is associated to the risk process

$$U(t) = U_1(t) + ... + U_m(t),$$

which, as in Proposition 2.4.2, is distributed the same way as the process

$$U'(t) = u + ct - \sum_{k=1}^{M'(t)} Y'_k - \sum_{k=1}^{N(t)} X'_k, \quad (5.2.6)$$

where $u = u_1 + ... + u_m$, $c = c_1 + ... + c_m$, $M'(t) = N_1(t) + ... + N_m(t)$ is a Poisson process with parameter $\lambda' = \lambda_1 + ... + \lambda_m$ and $\{X'_k\}_{k\geq 1}$, $\{Y'_k\}_{k\geq 1}$, $\{M'(t), t \geq 0\}$,

$\{N(t), t \geq 0\}$ are all mutually independent. Furthermore, $\{Y'_k\}_{k\geq 1}$ and $\{X'_k\}_{k\geq 1}$ respectively are independent and identically distributed random variables with distribution functions given by

$$F_{Y'}(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda'} F_{Y_i}(x) \quad \text{and} \quad F_{X'}(x) = F_{X_1+...+X_m}(x),$$

where $F_{Y_i}(x) = G_i(x) = P(Y_i \leq x)$ and $F_{X_1+...+X_m}(x) = P(X_1 + ... + X_m \leq x)$ represent the distribution functions of Y_i and $X_1 + ... + X_m$, respectively.

Hence, the process $U(t) = U_1(t) + ... + U_m(t)$ can be examined via $U'(t)$. The next subsection gives a review of the results related to the univariate process of the form (5.2.6), which can be applied for the ruin probability $\psi_{sum}(u)$.

In order to continue the search for results in a multivariate setting, we turn our attention to the other types of ruin probabilities from Definition 2.4.1.

By applying tools from the theory of PDM processes, we obtain martingales that are used in deriving bounds for ruin probabilities of type ψ_{sim}. Following discussions from Chapter 4 (for example, Remark 4.4.1), these martingale techniques are not successful in deriving results for ruin probability ψ_{and}.

As in Chapter 4, let us consider the case of light-tailed marginal claim size distributions.

For each $i = 1, ..., m$, $M_{X_i}(r_i)$ denotes the moment generating function (m.g.f.) of X_i. Therefore, assume that there exists $0 < r_i^0 \leq \infty$ such that $M_{X_i}(r_i) < \infty$ for all $r_i < r_i^0$ and $\lim_{r_i \uparrow r_i^0} M_{X_i}(r_i) = \infty$.

The joint moment generating function of $(X_1, ..., X_m)$ is defined by $M_{X_1,...,X_m}(r_1, ..., r_m) = E[e^{(r_1 X_1 + ... + r_m X_m)}]$. Clearly, $M_{X_1,...,X_m}(0, ..., 0) = 1$. Let $M = \{(r_1, .., r_m) \mid r_1, .., r_m \geq 0, M_{X_1,..,X_m}(r_1, .., r_m) < \infty\} - \{(0, .., 0)\}$.

As in Section 4.3, by the generalized Hölder's inequality (4.3.1) and the assumption $M_{X_i}(r_i) < \infty$ for $r_i < r_i^0$, if α_1,...,α_m are strictly positive numbers such that $\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_m} = 1$, then choosing $0 \leq r_i < r_i^0/\alpha_i$, $i = 1, ..., m$, with $(r_1, ..., r_m) \neq (0, ..., 0)$ implies

$$M_{X_1,...,X_m}(r_1, ..., r_m) \leq \prod_{i=1}^{m} \left(E[e^{\alpha_i r_i X_i}]\right)^{1/\alpha_i} = \prod_{i=1}^{m} [M_{X_i}(r_i \alpha_i)]^{1/\alpha_i} < \infty.$$

Therefore, the set M is non-empty.

Similar assumptions are made for the claim sizes Y_1,...,Y_m namely, for each $i = 1, ..., m$, let us assume that there exists $0 < r_i^1 \leq \infty$ such that $M_{Y_i}(r_i) < \infty$ for all $r_i < r_i^1$ and $\lim_{r_i \uparrow r_i^1} M_{Y_i}(r_i) = \infty$.

Consider now the set $R \subseteq [0, \infty)^m$ defined as

$$R = \{(r_1, .., r_m) | M_{X_1,...,X_m}(r_1, ..., r_m) < \infty, M_{Y_1}(r_1) < \infty, ..., M_{Y_m}(r_m) < \infty\}. \tag{5.2.7}$$

Under this setting, we proceed to derive upper bounds for the ruin probabilities ψ_{sim} associated to the models defined by (5.2.1) and (5.2.2), study that is presented in the next sections.

5.2.1. Review of the literature

This subsection is devoted to presenting some of the results related to the risk models defined by (5.2.1) and by (5.2.2), that are closely related to this chapter.

In Section 5.2, we mentioned that the problem involving the ruin probability of type ψ_{sum} can be reduced to a one-dimensional ruin problem defined by model (5.2.6) and hence, results concerning univariate versions of risk models defined by (5.2.1) or by (5.2.2) can be applied.

For example, the univariate risk model defined by (5.2.2) represents the Sparre Andersen risk model presented in Subsection 2.3.2. In addition to the results presented in Subsections 2.3.2 and 2.3.3, we want to emphasize a particular case of renewal process used in ruin problems namely, the Erlang (n) process which assumes that the distribution of the claim inter-arrival times $\{T_i\}_{i\geq1}$ is Erlang (n, $\widetilde{\lambda}$) with rate parameter $\widetilde{\lambda} > 0$, distribution that is described by Definition 1.1.1.

According to property (2) of Lemma 1.1.1, the expected inter-arrival time is $E[T] = n/\widetilde{\lambda}$.

The Sparre Andersen surplus process with Erlang (n, $\widetilde{\lambda}$) inter-arrival times $\{T_i\}_{i\geq1}$ has been widely studied in the literature, and in this sense, we mention Dickson (1998), Dickson and Hipp (1998, 2001), Li and Garrido (2004), Gerber and Shiu (2005), Li and Dickson (2006) and Li (2008). In what follows, we illustrate these papers by the following results.

For the univariate risk model (5.2.2), where $\{N(t), t \geq 0\}$ is an Erlang (n) process and the claim sizes $\{X_{1k}\}_{k\geq 1}$ are exponentially distributed with common mean μ_1, the net profit condition (5.2.5) is equivalent to

$$0 < c_1 - \frac{E[X_1]}{E[T]} = c_1 - \frac{\widetilde{\lambda}\mu_1}{n}.$$

From Dickson (1998) [see also Gerber and Shiu (2005), Li and Garrido (2004), Section 8], it follows that

$$\psi_1(u_1) = (1 - R\mu_1)e^{-Ru_1}, \tag{5.2.8}$$

where $-R < 0$ is the unique solution $r \in \mathbb{C}$ of Lundberg's fundamental equation for exponential claim sizes

$$\left(1 - \frac{c_1}{\widetilde{\lambda}}r\right)^n \left(\frac{1}{\mu_1} + r\right) - \frac{1}{\mu_1} = 0,$$

which is on the negative real line (all the other n solutions $r \in \mathbb{C}$ are in the right-half of the complex plane, see Gerber and Shiu (2005), Section 4). Note that $0 < R < \frac{1}{\mu_1}$ [formula (4.6) in Gerber and Shiu (2005) and Remark 1 in Li and Garrido (2004), p.395].

Result (5.2.8) is also given by Proposition 2.3.6, which gives the expression of the ruin probability for the general Sparre Andersen model where the claim sizes are exponentially distributed.

Next, associated to the univariate risk model (5.2.1), we mention the following results.

Yuen et al. (2002), for the univariate risk model of type (5.2.1) assuming that the common renewal process $N(t)$ is defined by inter-arrival times distributed as Erlang (2, $\widetilde{\lambda}$), obtained explicit expressions for the survival probability $\phi_1(u_1) = 1 - \psi_1(u_1)$ when the claim sizes are exponentially distributed. These results are illustrated by the following proposition.

Since the renewal claim inter-arrival times $\{T_i\}_{i\geq 1}$ follow an Erlang (2, $\widetilde{\lambda}$) distribution, according to Definition 1.1.1, the probability density function of T_i is $f(t) = \widetilde{\lambda}^2 t e^{-\widetilde{\lambda} t}$ for $t > 0$, or equivalently, $T_1 = T_{11} + T_{12}$, $T_2 = T_{21} + T_{22}$,...,where T_{11}, T_{12}, T_{21}, T_{22},... are independent exponential random variables with mean

$\widetilde{\lambda}^{-1}$, in view of property (5) of Lemma 1.1.1. In this case, the net profit condition (5.2.5) becomes

$$c_1 > \lambda_1 E[Y_1] + \tfrac{\widetilde{\lambda}}{2} E[X_1].$$

With other things being the same, consider the process obtained by setting T_1 equal to T_{12} instead of $T_{11} + T_{12}$. Therefore, T_1 is exponentially distributed with mean $\widetilde{\lambda}^{-1}$ and T_i still follows the same Erlang $(2, \widetilde{\lambda})$ distribution, for $i = 2, 3, ...$The corresponding survival probability for this process is denoted by $\phi^{(1)}(u_1)$, which is used in deriving $\phi_1(u_1)$. In this setup, Yuen et al. (2002) formulated the following result.

Proposition 5.2.1. *Consider the univariate risk model of type* (5.2.1)*, where* $\{N_1(t), t \geq 0\}$ *is a Poisson process with rate* λ_1 *and* $\{N(t), t \geq 0\}$ *is defined by inter-arrival times distributed as Erlang* $(2, \widetilde{\lambda})$*. If the claim sizes* X_1 *and* Y_1 *are exponentially distributed with equal mean* μ*, then the survival probability* $\phi_1(u_1)$ *is given by*

$$\phi_1(u_1) = 1 + C_2 A(z_2) e^{z_2 u_1} + C_3 A(z_3) e^{z_3 u_1},$$

where

$$A(z) = 1 + \frac{\mu}{\widetilde{\lambda}} \left(\lambda_1 + \widetilde{\lambda} - \frac{c_1}{\mu} \right) z - \frac{c_1 \mu z^2}{\widetilde{\lambda}},$$

z_2 *and* z_3 *are the two negative roots of the equation*

$$\frac{c_1^2 \mu}{\widetilde{\lambda}} z^3 - \frac{2 c_1 \mu}{\widetilde{\lambda}} \left[\lambda_1 + \widetilde{\lambda} - \frac{c_1}{\mu} \right] z^2 - \left[2c_1 - \frac{\mu}{\widetilde{\lambda}} (\lambda_1 + \widetilde{\lambda} - \frac{c_1}{\mu})^2 \right] z + 2\lambda_1 + \widetilde{\lambda} - \frac{2c_1}{\mu} = 0,$$

and the coefficients C_2 *and* C_3 *can be computed by the following equations*

$$\lambda_1 = \left[c_1 z_2 A(z_2) + \widetilde{\lambda} - (\lambda_1 + \widetilde{\lambda}) A(z_2) \right] C_2 + \left[c_1 z_3 A(z_3) + \widetilde{\lambda} - (\lambda_1 + \widetilde{\lambda}) A(z_3) \right] C_3,$$

$$\lambda_1 + \widetilde{\lambda} = (c_1 z_2 - \lambda_1 - \widetilde{\lambda}) C_2 + (c_1 z_3 - \lambda_1 - \widetilde{\lambda}) C_3.$$

Li and Garrido (2005) extended the result given by Proposition 5.2.1, by assuming that the process $\{N(t), t \geq 0\}$ is a renewal process with independent and identically distributed (i.i.d.) claim inter-arrival times $\{T_i\}_{i \geq 1}$ that are generalized Erlang (2) distributed, meaning that $T_i = T_{i1} + T_{i2}$ is defined as sum of two independent random variables, where $\{T_{i1}\}_{i \geq 1}$ are i.i.d. exponential random variables with mean $\widetilde{\lambda_1}^{-1}$, while the $\{T_{i2}\}_{i \geq 1}$ are i.i.d. exponential random variables with mean $\widetilde{\lambda_2}^{-1}$ ($\widetilde{\lambda_1}$ possibly different from $\widetilde{\lambda_2}$). In this case, the net profit condition (5.2.5) becomes

$$c_1 > \lambda_1\nu_1 + \frac{\widetilde{\lambda_1}\widetilde{\lambda_2}}{\widetilde{\lambda_1}+\widetilde{\lambda_2}}\mu_1,$$

providing a positive safety loading coefficient, θ_1, such that

$$\frac{1}{1+\theta_1} = \frac{1}{c_1}\left[\lambda_1\nu_1 + \frac{\widetilde{\lambda_1}\widetilde{\lambda_2}}{\widetilde{\lambda_1}+\widetilde{\lambda_2}}\mu_1\right]. \tag{5.2.9}$$

Also, if $g_1(x)$ and $f_1(x)$ denote the densities of Y_1 and X_1, respectively, then their Laplace transforms are denoted by $\widehat{g_1}(s) = \int_0^\infty e^{-sx}g_1(x)dx$ and $\widehat{f_1}(s) = \int_0^\infty e^{-sx}f_1(x)dx$.

Under this setting, Li and Garrido (2005) defined the Generalized Lundberg Fundamental Equation given by

$$\left[\frac{c_1}{\widetilde{\lambda_1}}s + \frac{\lambda_1}{\widetilde{\lambda_1}}(\widehat{g_1}(s)-1)-1\right] \times \left[\frac{c_1}{\widetilde{\lambda_2}}s + \frac{\lambda_1}{\widetilde{\lambda_2}}(\widehat{g_1}(s)-1)-1\right] = \widehat{f_1}(s), \quad s\in\mathbb{C}, \tag{5.2.10}$$

and showed the following result.

Proposition 5.2.2. *The generalized Lundberg equation in* (5.2.10) *has exactly one positive real root, say,* ρ.

With the aid of the solution ρ from Proposition 5.2.2, Li and Garrido (2005) derived the Laplace transform $\widehat{\phi_1}(s) = \int_0^\infty e^{-su_1}\phi_1(u_1)du_1$ of the survival probability $\phi_1(u_1)$, given by the following proposition.

Proposition 5.2.3. *Consider the univariate risk model of type* (5.2.1)*, where* $\{N_1(t), t\geq 0\}$ *is a Poisson process with rate* λ_1 *and* $\{N(t), t\geq 0\}$ *is a generalized Erlang (2) process. If* θ_1 *is given by* (5.2.9) *and* ρ *is the positive root of equation* (5.2.10)*, then the Laplace transform of* $\phi_1(u_1)$ *has the form*

$$\widehat{\phi_1}(s) = \frac{c_1\phi_1(0)\left\{c_1(s-\rho)+\lambda_1[\widehat{g_1}(s)-\widehat{g_1}(\rho)]\right\}}{\widetilde{\lambda_1}\widetilde{\lambda_2}[\gamma(s)-\widehat{f_1}(s)]},\ \ s\in\mathbb{C}, \tag{5.2.11}$$

where

$$\gamma(s) = \left[\frac{c_1}{\widetilde{\lambda_1}}s + \frac{\lambda_1}{\widetilde{\lambda_1}}(\widehat{g_1}(s)-1)-1\right] \times \left[\frac{c_1}{\widetilde{\lambda_2}}s + \frac{\lambda_1}{\widetilde{\lambda_2}}(\widehat{g_1}(s)-1)-1\right],$$

$$\textit{and}\ \ \phi_1(0) = \frac{\theta_1}{1+\theta_1} \times \frac{\widetilde{\lambda_1}+\widetilde{\lambda_2}}{c_1\rho - \lambda_1[1-\widehat{g_1}(\rho)]}. \tag{5.2.12}$$

In the case where both claims size distributions g_1 and f_1, belong to the class K_n class, $n \in \mathbb{N}^+$ [we refer to Willmot (1999)], meaning that

$$\widehat{g_1}(s) = \frac{\prod\limits_{i=1}^{n} a_i + h_1(s)}{\prod\limits_{i=1}^{n} (s + a_i)}, \; n \in \mathbb{N}^+,$$

$$\widehat{f_1}(s) = \frac{\prod\limits_{j=1}^{m} b_j + h_2(s)}{\prod\limits_{j=1}^{m} (s + b_j)}, \; m \in \mathbb{N}^+,$$

where $a_i > 0$, for $i = 1, 2, ..., n$, $b_j > 0$, for $j = 1, 2, ..., m$, $h_1(s)$ is a polynomial of degree $n - 1$ or less, $h_2(s)$ is a a polynomial of degree $m - 1$ or less with $h_1(0) = h_2(0) = 0$, Li and Garrido (2005) inverted the relationship (5.2.11) and obtained that the Laplace transform of $\phi_1(u_1)$ takes the following form:

$$\phi_1(u_1) = \phi_1(0) \left[C_0 + \sum_{i=1}^{2n+m} C_i e^{-R_i u_1} \right], \; u_1 > 0,$$

with $\phi_1(0)$ defined by (5.2.12). For the explicit expressions of the coefficients C_0, C_i and R_i $(i = 1, ..., 2n + m)$, we refer to Li and Garrido (2005).

In the general setting of the univariate risk model (5.2.1), where $\sum\limits_{k=1}^{N_1(t)} Y_{1k} + \sum\limits_{k=1}^{N(t)} X_{1k}$ is neither a compound renewal nor a compound Poisson process in general, Lv et al. (2010) derived an exponential upper bound for the ruin probability $\psi_1(u_1)$ using tools from the PDM processes theory. First, by using the backward Markovization technique, which is presented in Subsection 2.3.4, Lv et al. (2010) proved that the process $\{U_1(t), V(t), t \geq 0\}$, where $V(t) = t - \sigma_{N(t)}$ is the time elapsed since the last renewal claim arrival, is a Markov process with respect to the filtration $\mathcal{F}_t = \mathcal{F}_t^{U_1} \vee \mathcal{F}_t^{V}$. They also derived the infinitesimal generator associated to this Markov process and an exponential martingale, which are given by the following proposition.

Proposition 5.2.4. *Assume that for the univariate risk model* (5.2.1)*, the Laplace-Stieltjes transforms*

$$\widehat{G_1}(s)=\int_0^\infty e^{-sx}dG_1(x) \quad \textit{and} \quad \widehat{F_1}(s)=\int_0^\infty e^{-sx}dF_1(x)$$

of G_1 *and* F_1*, respectively exist and are twice differentiable on some interval* $[0,\alpha)$*, where* $\lim_{s\to\alpha}\widehat{G_1}(-s)=\infty$ *and* $\lim_{s\to\alpha}\widehat{F_1}(-s)=\infty$*. Then*

(1) The infinitesimal generator associated to the Markov process $\{U_1(t),V(t),t\geq 0\}$ *acting on a function* $f(z,v,t)$ *belonging to its domain is given by*

$$\begin{aligned}\mathcal{A}f(z,v,t)=&\frac{\partial f(z,v,t)}{\partial t}+c_1\frac{\partial f(z,v,t)}{\partial z}+\frac{\partial f(z,v,t)}{\partial v}\\&+\lambda_1\left[\int_0^\infty f(z-y,v,t)dG_1(y)-f(z,v,t)\right]\\&+\frac{q(v)}{1-Q(v)}\left[\int_{[0,\infty)} f(z-x,0,t)dF_1(x)-f(z,v,t)\right].\end{aligned}$$

(2) The process $\{g(V(t))e^{-\theta(r)t}e^{-rU_1(t)},t\geq 0\}$ *is a martingale where*

$$g(v)=\widehat{F_1}(-r)\frac{e^{\left[\theta(r)+c_1r-\lambda_1(\widehat{G_1}(-r)-1)\right]v}}{1-Q(v)}\int_v^\infty q(s)e^{-\left[\theta(r)+c_1r-\lambda_1(\widehat{G_1}(-r)-1)\right]s}ds,$$

and $\theta(r)$ *is determined by the following equation*

$$\widehat{F_1}(-r)\widehat{Q}\left(c_1r+\theta-\lambda_1(\widehat{G_1}(-r)-1)\right)=1. \tag{5.2.13}$$

Furthermore, using the forward Markovization technique (presented in Remark 2.3.2), Lv et al. (2010) obtained an exponential martingale based on the Markov process $\{U_1(t),W(t),t\geq 0\}$ with respect to the filtration $\mathcal{F}_t=\mathcal{F}_t^{U_1}\vee\mathcal{F}_t^{W}$, where $W(t)=\sigma_{N(t)+1}-t$ is the time remaining until the next renewal claim arrival. This martingale is given by the following proposition.

Proposition 5.2.5. *Assume that for the univariate risk model* (5.2.1)*, the Laplace-Stieltjes transforms*

$$\widehat{F_1}(s)=\int_0^\infty e^{-sx}dF_1(x) \quad \textit{and} \quad \widehat{G_1}(s)=\int_0^\infty e^{-sx}dG_1(x)$$

of F_1 *and* G_1*, respectively exist and are twice differentiable on some interval* $[0, \alpha)$*, where* $\lim_{s\to\alpha} \widehat{F_1}(-s) = \infty$ *and* $\lim_{s\to\alpha} \widehat{G_1}(-s) = \infty$*. Then the process* $\{g(W(t))e^{-\theta(r)t}e^{-rU_1(t)}, t \geq 0\}$ *is a martingale where*

$$g(w) = e^{-\left[\theta(r)+c_1r-\lambda_1(\widehat{G_1}(-r)-1)\right]w}, \tag{5.2.14}$$

and $\theta(r)$ *is determined by equation* (5.2.13)*.*

Based on the martingale obtained in Proposition 5.2.5, Lv et al. (2010) derived an upper bound of the ruin probability $\psi_1(u_1)$ as the following proposition shows.

Proposition 5.2.6. *Consider the univariate risk model of type* (5.2.1) *and let*

$$R = \sup\{r \geq 0 : \widehat{F_1}(-r)\widehat{Q}(c_1r - \lambda_1(\widehat{G_1}(-r) - 1)) \leq 1\}.$$

If $B_{rc_1} = \sup\{E[T - x|T > x] : x \geq 0\} < \infty$*, then the ruin probability satisfies*

$$\psi_1(u_1) \leq C(R)e^{-Ru_1},$$

where $C(R)$ *is a finite constant defined as*

$$C(R) = E[\max\{g(W(0))e^{(c_1r+\theta(r)-\lambda_1(\widehat{G_1}(-r)-1))B_{rc_1}}, 1\}],$$

and $\theta(r)$ *is determined by equation* (5.2.13) *and* $g(w)$ *by* (5.2.14)*.*

Regarding models (5.2.1) or (5.2.2) in a multivariate setting, we conclude this section with two results concerning the asymptotic behavior of the finite-time ruin probabilities of types ψ_{or} and ψ_{and} associated to the bivariate risk model (5.2.2), where the claims distributions belong to the consistent variation class. These results were formulated by Chen et al. (2011) and are given in the following proposition.

Proposition 5.2.7. *For the bivariate risk model* (5.2.2) *with the net profit condition* (5.2.5) *fulfilled for each class of business, suppose that the random variables* X_{1k} *and* X_{2k} *are independent for each* $k \geq 1$*, that their distributions* F_1 *and* F_2 *belong to class* $\mathcal{C}$ *and that* $E[T_1^p] < \infty$ *for some* $p > J_{F_1}^+ + J_{F_2}^+ + 1$ *(see Definition* 2.3.6 *for the index* J_F^+*). Then, it holds uniformly for all* $t \in \Lambda = \{t : m(t) > 0\}$

as $(u_1, u_2) \to (\infty, \infty)$ *that*

$$\psi_{and}(u_1, u_2, t) \sim E\left[\prod_{j=1}^{2} \frac{\lambda}{c_j - \lambda\mu_j} \int\limits_{u_j}^{u_j + \frac{c_j - \lambda\mu_j}{\lambda} N(t)} [1 - F_j(y)]dy\right],$$

and that

$$\psi_{or}(u_1, u_2, t) \sim \sum_{j=1}^{2} \frac{\lambda}{c_j - \lambda\mu_j} \int\limits_{u_j}^{u_j + \frac{c_j - \lambda\mu_j}{\lambda} m(t)} [1 - F_j(y)]dy. \tag{5.2.15}$$

We point out that the result (5.2.15) is an extension of the result (2.3.21) from Proposition 2.3.16 concerning the univariate renewal model.

5.3. The backward Markovization technique and martingales

Our goal is to obtain exponential martingales, which will be used in the ruin problem associated with the risk model in (5.2.1). For this, Proposition 2.1.5, which gives a connection between martingales and Markov processes, will be applied.

Therefore, we first need to identify a Markov process, compute its infinitesimal generator and then obtain the martingale via Proposition 2.1.5. These steps are illustrated by the propositions of this section.

Due to the presence of the renewal claim process $\{N(t), t \geq 0\}$ in the multivariate risk model (5.2.1), for which the time since the last claim provides information on the time until the next claim occurs, the surplus vector process $\mathbf{U}(t)$ is not a Markov process. As in the univariate renewal model, presented in Subsection 2.3.2, the surplus vector process $\mathbf{U}(t)$ can be made Markovian by introducing a supplementary process: $V(t) = t - \sigma_{N(t)}$, which represents the time elapsed since the last renewal claim before time t. This technique, called backward Markovization technique, can be found in Cox (1955).

Assume that the vector process $\{(\mathbf{U}(t), V(t)), t \geq 0\}$ is defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with state space $(\mathbb{R}^m \times \mathbb{R}_+, \mathfrak{B}(\mathbb{R}^m) \times \mathfrak{B}(\mathbb{R}_+))$. Here, $\mathcal{F}_t = \mathcal{F}_t^{U_1} \vee ... \vee \mathcal{F}_t^{U_m} \vee \mathcal{F}_t^{V}$, where $\mathcal{F}_t^{U_i}$, for $1 \leq i \leq m$, and $\mathcal{F}_t^{V}$ are the natural filtrations, introduced by Definition 2.1.3, of the processes $\{U_i(t), t \geq 0\}$ and $\{V(t), t \geq 0\}$, respectively.

As in the univariate case studied by Lv et al. (2010), the following proposition establishes the Markov property of the vector process $\{(\mathbf{U}(t), V(t)), t \geq 0\}$.

Proposition 5.3.1. *If $\boldsymbol{U}(t)$ is defined by relation* (5.2.1) *and $V(t) = t - \sigma_{N(t)}$, then the process $\{(\boldsymbol{U}(t), V(t)), t \geq 0\}$ is a Markov vector process.*

Proof. Following Definition 2.1.5, we need to prove that

$$P\left(\mathbf{U}(t+h) \in A, V(t+h) \in B|\mathcal{F}_t\right)$$

$$= P\left(\mathbf{U}(t+h) \in A, V(t+h) \in B|\mathbf{U}(t), V(t)\right),$$

for all t, $h \geq 0$ and Borel sets $A \in \mathfrak{B}(\mathbb{R}^m)$ and $B \in \mathfrak{B}(\mathbb{R}_+)$. For this, it is enough to show that $P\left(\mathbf{U}(t+h) > \mathbf{a}, V(t+h) > b|\mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F\right)$

$$= P\left(\mathbf{U}(t+h) > \mathbf{a}, V(t+h) > b|\mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0\right), \tag{5.3.1}$$

for any $F \in \mathcal{F}_t$, $\mathbf{a}$, $\mathbf{a}_0 \in \mathbb{R}^m$ and b, $b_0 \geq 0$. We may assume that $F = (\mathbf{U}(s) > \mathbf{a}_1, V(s) > b_1)$ for $0 \leq s < t$, $\mathbf{a}_1 \in \mathbb{R}^m$ and $b_1 \geq 0$. The left-hand side of relation (5.3.1) is equal to

$$\frac{P\left(\mathbf{U}(t+h) > \mathbf{a}, V(t+h) > b, \mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F\right)}{P\left(\mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F\right)}. \tag{5.3.2}$$

Therefore, we start by computing:

$$P\left(\mathbf{U}(t+h) > \mathbf{a}, V(t+h) > b, \mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F\right)$$

$$= P(\mathbf{U}(t) + \mathbf{c}h - \sum_{\mathbf{i}=\mathbf{M}(t)+\mathbf{1}}^{\mathbf{M}(t+h)} \mathbf{Y_i} - \sum_{i=N(t)+1}^{N(t+h)} \mathbf{X}_i > \mathbf{a}, t+h-\sigma_{N(t+h)} > b,$$
$$\mathbf{U}(t) = \mathbf{a}_0, t - \sigma_{N(t)} = b_0, \mathbf{U}(s) > \mathbf{a}_1, V(s) > b_1) \tag{5.3.3}$$

$$= \sum_{\mathbf{J},\mathbf{K}} \sum_{\mathbf{L}\leq\mathbf{J}} \sum_{j,k} \sum_{l\leq j} P\left(\mathbf{c}h - \sum_{\mathbf{i}=\mathbf{J}+\mathbf{1}}^{\mathbf{J}+\mathbf{K}} \mathbf{Y_i} - \sum_{i=j+1}^{j+k} \mathbf{X}_i > \mathbf{a} - \mathbf{a}_0, h - \sum_{i=j+1}^{j+k} T_i > b - b_0, C\right), \tag{5.3.4}$$

where

$$\mathbf{i} = \underbrace{(i, ..., i)}_{m},\ \mathbf{J} = (j_1, ..., j_m),\ \mathbf{K} = (k_1, ..., k_m),\ \mathbf{L} = (l_1, ..., l_m),\ \mathbf{1} = \underbrace{(1, ..., 1)}_{m},$$

$$\sum_{\mathbf{i}=\mathbf{J}+\mathbf{1}}^{\mathbf{J}+\mathbf{K}} \mathbf{Y}_{\mathbf{i}} = \left(\sum_{i=j_1+1}^{j_1+k_1} Y_{1i}, ..., \sum_{i=j_m+1}^{j_m+k_m} Y_{mi} \right), \quad \sum_{i=j+1}^{j+k} \mathbf{X}_i = \left(\sum_{i=j+1}^{j+k} X_{1i}, ..., \sum_{i=j+1}^{j+k} X_{mi} \right),$$

and $C = \{C_1, C_2, C_3\}$ with

$$C_1 = (\mathbf{M}(t+h) - \mathbf{M}(t) = \mathbf{K},\ N(t+h) - N(t) = k),$$

$$C_2 = \left(\mathbf{M}(t) = \mathbf{J}, N(t) = j, \mathbf{u} + \mathbf{c}t - \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{J}} \mathbf{Y}_{\mathbf{i}} - \sum_{i=1}^{j} \mathbf{X}_i = \mathbf{a}_0, t - \sigma_j = b_0 \right),$$

$$C_3 = \left(\mathbf{M}(s) = \mathbf{L}, N(s) = l, \mathbf{u} + \mathbf{c}s - \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{L}} \mathbf{Y}_{\mathbf{i}} - \sum_{i=1}^{l} \mathbf{X}_i > \mathbf{a}_1, s - \sigma_l > b_1 \right).$$

To establish the last equality given by relation (5.3.4), we use the fact that the event of (5.3.3) is the union of disjoint events of type

$$\left(\mathbf{c}h - \sum_{\mathbf{i}=\mathbf{J}+\mathbf{1}}^{\mathbf{J}+\mathbf{K}} \mathbf{Y}_{\mathbf{i}} - \sum_{i=j+1}^{j+k} \mathbf{X}_i > \mathbf{a} - \mathbf{a}_0, h - \sum_{i=j+1}^{j+k} T_i > b - b_0, C \right),$$

where j_1,...j_m, k_1,...,k_m, l_1,...,l_m, j, k, l run over the positive integers with $l_i \leq j_i$ $(1 \leq i \leq m)$ and $l \leq j$.

By using the property that the Poisson processes $N_i(t)$, $i = 1, ..., m$, have independent and stationary increments and the hypothesis that $\{Y_{1i}\}_{i\geq 1}$,...,$\{Y_{mi}\}_{i\geq 1}$, $\{\mathbf{X}_i\}_{i\geq 1} = \{(X_{1i}, ..., X_{mi})\}_{i\geq 1}$, $N_1(t)$,...,$N_m(t)$, $N(t)$ are all mutually independent, relation (5.3.4) becomes

$$\sum_{\mathbf{J},\mathbf{K}} \sum_{\mathbf{L}\leq\mathbf{J}} \sum_{j,k} \sum_{l\leq j} P \left(\mathbf{c}h - \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{K}} \mathbf{Y}_{\mathbf{i}} - \sum_{i=1}^{k} \mathbf{X}_i > \mathbf{a} - \mathbf{a}_0 \right)$$

$$\times P\left(\mathbf{M}(h) = \mathbf{K}\right) (P(h - \sum_{i=j+1}^{j+k} T_i > b - b_0,$$

$$N(t+h) - N(t) = k | N(t) = j, t - \sigma_j = b_0, N(s) = l, s - \sigma_l > b_1) P\left(C_2 C_3\right)$$

$$= \sum_{\mathbf{J},\mathbf{K}} \sum_{\mathbf{L}\leq\mathbf{J}} \sum_{j,k} \sum_{l\leq j} P \left(\mathbf{c}h - \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{K}} \mathbf{Y}_{\mathbf{i}} - \sum_{i=1}^{k} \mathbf{X}_i > \mathbf{a} - \mathbf{a}_0 \right)$$

$$\times P\left(\mathbf{M}(h) = \mathbf{K}\right) P(h - \sum_{i=j+1}^{j+k} T_i > b - b_0,$$

$$\sum_{i=j+1}^{j+k} T_i - b_0 \leq h < \sum_{i=j+1}^{j+k+1} T_i - b_0 | N(t) = j, t - \sigma_j = b_0, N(s) = l, s - \sigma_l > b_1) P\left(C_2 C_3\right), \tag{5.3.5}$$

where for the last step, relation (2.2.1) is used. Now, for the term

$$P(h-\sum_{i=j+1}^{j+k}T_i>b-b_0,\sum_{i=j+1}^{j+k}T_i-b_0\le h<\sum_{i=j+1}^{j+k+1}T_i-b_0|N(t)=j,$$
$$t-\sigma_j=b_0,N(s)=l,s-\sigma_l>b_1)$$

of (5.3.5), we write

$$(N(t)=j,t-\sigma_j=b_0,N(s)=l,s-\sigma_l>b_1)$$
$$=\left(\sum_{i=1}^{j}T_i\le t<\sum_{i=1}^{j+1}T_i,t-\sum_{i=1}^{j}T_i=b_0,\sum_{i=1}^{l}T_i\le s<\sum_{i=1}^{l+1}T_i,s-\sum_{i=1}^{l}T_i>b_1\right)$$
$$=\left(T_{j+1}>b_0,t-\sum_{i=1}^{j}T_i=b_0,\sum_{i=1}^{l}T_i\le s<\sum_{i=1}^{l+1}T_i,s-\sum_{i=1}^{l}T_i>b_1\right),$$

and we consider the following two cases:

Case 1: If $l<j$, then $l+1\le j<j+1$.

Case 2: If $l=j$, then
$$\left(T_{j+1}>b_0,t-\sum_{i=1}^{j}T_i=b_0,\sum_{i=1}^{j}T_i\le s<\sum_{i=1}^{j+1}T_i,s-\sum_{i=1}^{j}T_i>b_1\right)$$
$$=\left(T_{j+1}>b_0,t-\sum_{i=1}^{j}T_i=b_0,s-\sum_{i=1}^{j}T_i>b_1\right).$$
Indeed, the inclusion " $\subseteq$ " is obvious and " $\supseteq$ " is established by the following: $\sum_{i=1}^{j}T_i<s-b_1<s,$ and $\sum_{i=1}^{j+1}T_i=T_{j+1}+t-b_0>b_0+t-b_0>s.$

By considering these two cases, it results that the event

$$(N(t)=j,t-\sigma_j=b_0,N(s)=l,s-\sigma_l>b_1)$$

depends on T_{j+1} only through the event $(T_{j+1}>b_0)$.

Combining these two cases with the assumption that $\{T_i\}_{i\ge1}$ are independent and identically distributed random variables, we have that (5.3.5) is equal to

$$\sum_{\mathbf{J},\mathbf{K}}\sum_{\mathbf{L}\le\mathbf{J}}\sum_{j,k}\sum_{l\le j}P\left(\mathbf{c}h-\sum_{\mathbf{i=1}}^{\mathbf{K}}\mathbf{Y_i}-\sum_{i=1}^{k}\mathbf{X}_i>\mathbf{a}-\mathbf{a}_0\right)$$
$$\times P\left(\mathbf{M}(h)=\mathbf{K}\right)P(h-\sum_{i=j+1}^{j+k}T_i>b-b_0,$$
$$\sum_{i=j+1}^{j+k}T_i-b_0\le h<\sum_{i=j+1}^{j+k+1}T_i-b_0|T_{j+1}>b_0)P\left(C_2C_3\right)$$
$$=\sum_{\mathbf{K},k}P\left(\mathbf{c}h-\sum_{\mathbf{i=1}}^{\mathbf{K}}\mathbf{Y_i}-\sum_{i=1}^{k}\mathbf{X}_i>\mathbf{a}-\mathbf{a}_0\right)$$
$$\times P\left(\mathbf{M}(h)=\mathbf{K}\right)P(h-\sum_{i=1}^{k}T_i>b-b_0,$$

$$\sum_{i=1}^{k} T_i \leq b_0 + h < \sum_{i=1}^{k+1} T_i | T_1 > b_0) \sum_{\mathbf{J},j} \sum_{\mathbf{L} \leq \mathbf{J}} \sum_{l \leq j} P\left(C_2 C_3\right)$$

$$= \sum_{\mathbf{K},k} P\left(\mathbf{c}h - \sum_{\mathbf{i=1}}^{\mathbf{K}} \mathbf{Y_i} - \sum_{i=1}^{k} \mathbf{X}_i > \mathbf{a} - \mathbf{a}_0\right) P\left(\mathbf{M}(h) = \mathbf{K}\right)$$

$$\times P\left(h - \sum_{i=1}^{k} T_i > b - b_0, \sum_{i=1}^{k} T_i \leq b_0 + h < \sum_{i=1}^{k+1} T_i | T_1 > b_0\right)$$

$$\times P\left(\mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F\right), \tag{5.3.6}$$

where in the last step we use the fact that the event $(\mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F)$ is the union of disjoint events of the form (C_2C_3). Therefore, plugging (5.3.6) into (5.3.2), the left-hand side of (5.3.1) becomes

$$P\left(\mathbf{U}(t+h) > \mathbf{a}, V(t+h) > b | \mathbf{U}(t) = \mathbf{a}_0, V(t) = b_0, F\right)$$

$$= \sum_{\mathbf{K},k} P\left(\mathbf{c}h - \sum_{\mathbf{i=1}}^{\mathbf{K}} \mathbf{Y_i} - \sum_{i=1}^{k} \mathbf{X}_i > \mathbf{a} - \mathbf{a}_0\right) P\left(\mathbf{M}(h) = \mathbf{K}\right)$$

$$\times P\left(h - \sum_{i=1}^{k} T_i > b - b_0, \sum_{i=1}^{k} T_i \leq b_0 + h < \sum_{i=1}^{k+1} T_i | T_1 > b_0\right). \tag{5.3.7}$$

The right-hand side of (5.3.1) is computed in a similar manner and is equal to the right-hand side of (5.3.7) and hence, the proof is complete. □

An immediate consequence of this proposition follows for the special case of the common shocks only.

Corollary 5.3.1. *If $\boldsymbol{U}(t)$ is defined by relation* (5.2.2) *and $V(t) = t - \sigma_{N(t)}$, then the process $\{(\boldsymbol{U}(t), V(t)), t \geq 0\}$ is a Markov vector process.*

The next step is to derive the infinitesimal generator associated to the Markov process established in Proposition 5.3.1.

First, we consider the special case given by (5.2.2), where we assume that the claims occur only due to common shocks, and hence, $N_i(t) \equiv 0$ for $i = 1, ..., m$. The reason for considering this case is to show that the infinitesimal generator can be computed by a direct application of Proposition 2.1.2 from the theory of PDM processes. This result is followed by deriving the infinitesimal generator in the more general case given by (5.2.1).

Theorem 5.3.1. *Consider the m-dimensional risk model defined by* (5.2.2)*. Then the infinitesimal generator of the homogeneous Markov process* $\{(\boldsymbol{U}(t), V(t), t), t \geq 0\}$ *acting on a function* $f(\boldsymbol{z}, v, t)$ *belonging to its domain is given by*

$$\mathcal{A}f(\boldsymbol{z}, v, t) = \frac{\partial f(\boldsymbol{z}, v, t)}{\partial t} + \sum_{i=1}^{m} c_i \frac{\partial f(\boldsymbol{z}, v, t)}{\partial z_i} + \frac{\partial f(\boldsymbol{z}, v, t)}{\partial v}$$
$$+ \frac{q(v)}{1 - Q(v)} \left[\int\limits_{[0,\infty)^m} f(\boldsymbol{z} - \boldsymbol{x}, 0, t) dF(\boldsymbol{x}) - f(\boldsymbol{z}, v, t) \right], \tag{5.3.8}$$

where $(\boldsymbol{z}, v, t) = (z_1, ..., z_m, v, t) \in \mathbb{R}^m \times \mathbb{R}_+^2$ *and* $f : \mathbb{R}^m \times \mathbb{R}_+^2 \to (0, \infty)$ *is differentiable with respect to* $z_1, ..., z_m, v, t$ *for all* $(z_1, ..., z_m, v, t) \in \mathbb{R}^m \times \mathbb{R}_+^2$.

Proof. In between jumps the Markov vector process $(\mathbf{U}(t), V(t))$ with the state space $E = \mathbb{R}^m \times \mathbb{R}_+$ evolves deterministically as $\mathbf{U}(t) = \mathbf{z} + \mathbf{c}t$, $V(t) = v + t$, for some values $\mathbf{z} \in \mathbb{R}^m$, $v \geq 0$, and

$$\tfrac{d}{dt} f(\mathbf{z} + \mathbf{c}t, v + t) = \sum_{i=1}^{m} c_i \tfrac{\partial}{\partial z_i} f(\mathbf{z} + \mathbf{c}t, v + t) + \tfrac{\partial}{\partial v} f(\mathbf{z} + \mathbf{c}t, v + t),$$

for $f : \mathbb{R}^m \times \mathbb{R}_+ \to (0, \infty)$. Thus, the differential operator is

$$\chi = \sum_{i=1}^{m} c_i \frac{\partial}{\partial z_i} + \frac{\partial}{\partial v}.$$

At the jump times σ_i, we have $\mathbf{U}(\sigma_i) = \mathbf{U}(\sigma_i -) - \mathbf{X}$, $V(\sigma_i) = 0$, and the jump intensity is given by the hazard rate function of the inter-arrival times, that is,

$$\lambda(v) = \lim_{h \downarrow 0} \frac{P\left(T_i \in (v, v + h) | T_i > v\right)}{h} = \frac{q(v)}{1 - Q(v)}.$$

Applying Proposition 2.1.2 leads to expression (5.3.8) of the infinitesimal generator of the process $(\mathbf{U}(t), V(t), t)$.

Since the domain of the generator $\mathcal{A}$, $D(\mathcal{A})$, consists of all measurable functions for which the expression of $\mathcal{A}(f)$ exists, for $f(z_1, ..., z_m, v, t)$ to belong to $D(\mathcal{A})$, it is sufficient that $f(z_1, ..., z_m, v, t)$ be differentiable with respect to $z_1,...,z_m$, v, t for all $z_1,...,z_m$, v, t and that

$$\left| \int\limits_{[0,\infty)^m} f(\mathbf{z} - \mathbf{x}, 0, t) dF(\mathbf{x}) - f(\mathbf{z}, v, t) \right| < \infty.$$

□

Theorem 5.3.2. *Consider the m-dimensional risk model defined by* (5.2.1). *Then the infinitesimal generator of the homogeneous Markov process* $\{(\boldsymbol{U}(t), V(t), t), t \geq 0\}$ *acting on a function* $f(\boldsymbol{z}, v, t)$ *belonging to its domain is given by*

$$\mathcal{A}f(\boldsymbol{z}, v, t) = \frac{\partial f(\boldsymbol{z}, v, t)}{\partial t} + \sum_{i=1}^{m} c_i \frac{\partial f(\boldsymbol{z}, v, t)}{\partial z_i} + \frac{\partial f(\boldsymbol{z}, v, t)}{\partial v}$$

$$+ \sum_{i=1}^{m} \lambda_i \left[\int_0^{\infty} f(z_1, ..., z_i - y_i, ..., z_m, v, t) dG_i(y_i) - f(z_1, ..., z_m, v, t) \right]$$

$$+ \frac{q(v)}{1 - Q(v)} \left[\int_{[0,\infty)^m} f(\boldsymbol{z} - \boldsymbol{x}, 0, t) dF(\boldsymbol{x}) - f(\boldsymbol{z}, v, t) \right], \qquad (5.3.9)$$

where $(\boldsymbol{z}, v, t) = (z_1, ..., z_m, v, t) \in \mathbb{R}^m \times \mathbb{R}_+^2$ *and* $f : \mathbb{R}^m \times \mathbb{R}_+^2 \to (0, \infty)$ *is differentiable with respect to* $z_1, ..., z_m, v, t$ *for all* $(z_1, ..., z_m, v, t) \in \mathbb{R}^m \times \mathbb{R}_+^2$.

PROOF. By Definition 2.1.6, we have

$$\mathcal{A}f(\mathbf{z}, v, t) =$$

$$\lim_{h \downarrow 0} \frac{E[f(\mathbf{U}(t+h), V(t+h), t+h) \mid \mathbf{U}(t) = \mathbf{z}, V(t) = v, t] - f(\mathbf{z}, v, t)}{h}, \quad (5.3.10)$$

where the domain of $\mathcal{A}$ is the set of all measurable functions f for which this limit exists. We start by computing

$$E[f(\mathbf{U}(t+h), V(t+h), t+h) \mid \mathbf{U}(t) = \mathbf{z}, V(t) = v, t]$$

$$= E[f(\mathbf{z} + \mathbf{c}h - \sum_{\mathbf{k}=\mathbf{M}(t)+\mathbf{1}}^{\mathbf{M}(t+h)} \mathbf{Y_k} - \sum_{k=N(t)+1}^{N(t+h)} \mathbf{X}_k,$$

$$v + h - \sum_{k=N(t)+1}^{N(t+h)} T_k, t+h) \mid \mathbf{U}(t) = \mathbf{z}, V(t) = v, t]. \qquad (5.3.11)$$

For the small time interval $(t, t+h]$, we consider the following possible cases:

(1) no claim occurs in $(t, t+h]$,

(2) only one claim occurs in $(t, t+h]$, and

(3) more than one claim occurs in $(t, t+h]$.

By virtue of property (2) given by Proposition 2.2.4, the probability of the event in case (3) is $o(h)$. Since $N_1(t)$,...,$N_m(t)$, $N(t)$ are all mutually independent, the inter-arrival times $\{T_i\}_{i\geq 1}$ are independent and identically distributed random variables,

$$(V(t) = t - \sum_{k=1}^{N(t)} T_k = v) = (T_{N(t)+1} > v),$$

and by the total law of probability, (5.3.11) is equal to

$$\tfrac{P(T_1>v+h)}{P(T_1>v)}\ e^{-h\sum\limits_{i=1}^{m}\lambda_i} f(\mathbf{z}+\mathbf{c}h, v+h, t+h)$$

$$+\tfrac{P(T_1>v+h)}{P(T_1>v)} \sum_{i=1}^{m} \lambda_i h e^{-h\sum\limits_{i=1}^{m}\lambda_i} \int_0^\infty f(z_1{+}c_1h, ., z_i{+}c_ih{-}y_i, ., z_m{+}c_mh, v{+}h, t{+}h) dG_i(y_i)$$

$$+\tfrac{e^{-h\sum\limits_{i=1}^{m}\lambda_i}}{P(T_1>v)} \int\limits_{[0,\infty)^m} \int\limits_v^{v+h} f(\mathbf{z}{+}\mathbf{c}h{-}\mathbf{x}, v{+}h{-}v_1, t{+}h) P(T_2 > v{+}h{-}v_1) q(v_1) dv_1 dF(\mathbf{x})$$

$$+o(h),$$

where the first term corresponds to case (1) determined by the event

$$(N(t+h) - N(t) = 0, N_i(t+h) - N_i(t) = 0, i = 1, ..., m)\,.$$

The next two terms correspond to case (2) determined by the union of the following disjoint events for $i = 1, ..., m$:

$$(N(t+h) - N(t) = 0, N_1(t+h) - N_1(t) = 1, N_i(t+h) - N_i(t) = 0, i \neq 1)\,,$$

$$.$$

$$.$$

$$.$$

$$(N(t+h) - N(t) = 0, N_m(t+h) - N_m(t) = 1, N_i(t+h) - N_i(t) = 0, i \neq m)\,,$$

$$(N(t+h) - N(t) = 1, N_i(t+h) - N_i(t) = 0, i = 1, ..., m)\,.$$

The last term corresponds to case (3), and result (2) of Proposition 2.2.4 was applied. Now, using this result in (5.3.10), we further obtain

$$\mathcal{A}f(\mathbf{z}, v, t) = \lim_{h\downarrow 0} \frac{f(\mathbf{z}+\mathbf{c}h, v+h, t+h) - f(\mathbf{z}, v, t)}{h}$$

$$+\lim_{h\downarrow 0}\frac{e^{-h\sum\limits_{i=1}^{m}\lambda_i}-1}{h}f(\mathbf{z}+\mathbf{c}h,v+h,t+h)$$

$$+\sum_{i=1}^{m}\lambda_i\int_0^\infty f(z_1,...,z_i-y_i,...,z_m,v,t)dG_i(y_i)$$

$$+\frac{1}{1-Q(v)}$$

$$\times\lim_{h\downarrow 0}\int\limits_{[0,\infty)^m}\frac{\int\limits_v^{v+h}f(\mathbf{z}+\mathbf{c}h-\mathbf{x},v+h-v_1,t+h)\overline{Q}(v+h-v_1)q(v_1)dv_1}{h}dF(\mathbf{x}).$$

(5.3.12)

For the first limit of (5.3.12), using the Taylor series' expansion leads to the following result

$$\frac{\partial f(\mathbf{z},v,t)}{\partial t}+\sum_{i=1}^{m}c_i\frac{\partial f(\mathbf{z},v,t)}{\partial z_i}+\frac{\partial f(\mathbf{z},v,t)}{\partial v}.$$

The second limit of (5.3.12) is equal to

$$-\sum_{i=1}^{m}\lambda_i f(\mathbf{z},v,t),$$

and for the third limit, the Mean Value Theorem is used and yields

$$+\frac{q(v)}{1-Q(v)}\int\limits_{[0,\infty)^m}f(\mathbf{z}-\mathbf{x},0,t)dF(\mathbf{x}).$$

Consequently, result (5.3.9) of this proposition is established.

To complete the proof, note that for $f(z_1,...,z_m,v,t)$ to belong to the domain of the generator $\mathcal{A}$, it is sufficient that $f(z_1,...,z_m,v,t)$ be differentiable with respect to z_1,...,z_m, v, t for all z_1,...,z_m, v, t, and that

$$\left|\int_0^\infty f(z_1,...,z_i-y_i,...,z_m,v,t)dG_i(y_i)-f(z_1,...,z_m,v,t)\right|<\infty,\ i=1,..,m, \tag{5.3.13}$$

and $$\left|\int\limits_{[0,\infty)^m}f(\mathbf{z}-\mathbf{x},0,t)dF(\mathbf{x})-f(\mathbf{z},v,t)\right|<\infty. \tag{5.3.14}$$

□

The following theorem gives the construction of an exponential martingale, which will be used in our ruin related problem presented in the next section.

For simplicity, $M(r_1, ..., r_m)$ stands for $M_{X_1,...,X_m}(r_1, ..., r_m)$. Also, let us define

$$h_\theta(r_1, ..., r_m) = M(r_1, ..., r_m) \int_0^\infty q(x) e^{-\left[\theta + \sum_{i=1}^m \left(c_i r_i - \lambda_i M_{Y_i}(r_i) + \lambda_i\right)\right] x} dx - 1. \quad (5.3.15)$$

Theorem 5.3.3. *Consider the risk model* (5.2.1) *and the function* h_θ *defined by* (5.3.15) *such that* $\sup_{(r_1,...,r_m)\in R} h_\theta(r_1, ..., r_m) > 0$*, where the set* R *is given by* (5.2.7)*. Then for every* $\theta \geq 0$*, the process*

$$Z_\theta(t) = k(V(t)) e^{-\theta t} e^{-r_{1\theta} U_1(t) - ... - r_{m\theta} U_m(t)}, \; t \geq 0, \quad (5.3.16)$$

is a martingale with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$*, where*

$$k(v) = \frac{e^{\left[\theta + \sum_{i=1}^m \left(c_i r_{i\theta} - \lambda_i M_{Y_i}(r_{i\theta}) + \lambda_i\right)\right] v} M(r_{1\theta}, ., r_{m\theta})}{1 - Q(v)}$$
$$\times \int_v^\infty q(x) e^{-\left[\theta + \sum_{i=1}^m \left(c_i r_{i\theta} - \lambda_i M_{Y_i}(r_{i\theta}) + \lambda_i\right)\right] x} dx, \quad (5.3.17)$$

and $(r_{1\theta}, ..., r_{m\theta}) \in R$ *is determined by the equation*

$$h_\theta(r_1, ..., r_m) = 0. \quad (5.3.18)$$

Proof. Proposition 2.1.5 implies that for a function f belonging to the domain of the infinitesimal generator described by relation (5.3.9) such that $\mathcal{A}f = 0$, the process $\{f(\mathbf{U}(t), V(t), t), t \geq 0\}$ is a martingale. The equation $\mathcal{A}f = 0$ is equivalent to

$$\frac{\partial f(\mathbf{z}, v, t)}{\partial t} + \sum_{i=1}^m c_i \frac{\partial f(\mathbf{z}, v, t)}{\partial z_i} + \frac{\partial f(\mathbf{z}, v, t)}{\partial v}$$
$$+ \sum_{i=1}^m \lambda_i \left[\int_0^\infty f(z_1, ..., z_i - y_i, ..., z_m, v, t) dG_i(y_i) - f(z_1, ..., z_m, v, t) \right]$$
$$+ \frac{q(v)}{1 - Q(v)} \left[\int_{[0,\infty)^m} f(\mathbf{z} - \mathbf{x}, 0, t) dF(\mathbf{x}) - f(\mathbf{z}, v, t) \right] = 0. \quad (5.3.19)$$

In fact, we need only a special solution. Intuitively, an exponential form of the solution is suitable for this equation and convenient for the conditions (5.3.13) and (5.3.14) to be satisfied. As in Dassios and Embrechts (1989), assume a solution of the form $f(\mathbf{z}, v, t) = k(v)e^{-\theta t}e^{-r_1 z_1 - \ldots - r_m z_m}$, where $\theta \geq 0$, $(r_1, ..., r_m) \in R$, $k(v)$ is differentiable in v, and we may also assume that $k(0) = 1$. Substituting this form in equation (5.3.19), we obtain

$$-\sum_{i=1}^{m} c_i r_i k(v) - \theta k(v) + k'(v) + \sum_{i=1}^{m} \lambda_i \left[M_{Y_i}(r_i) - 1\right] k(v)$$

$$+\frac{q(v)}{1-Q(v)} M(r_1, ..., r_m) - \frac{q(v)}{1-Q(v)} k(v) = 0,$$

and further, it follows that

$$\frac{d}{dv}\left[k(v)(1-Q(v))e^{-\left(\theta + \sum\limits_{i=1}^{m} c_i r_i - \sum\limits_{i=1}^{m} \lambda_i \left(M_{Y_i}(r_i) - 1\right)\right)v}\right]$$

$$= -M(r_1, ..., r_m) q(v) e^{-\left(\theta + \sum\limits_{i=1}^{m} c_i r_i - \sum\limits_{i=1}^{m} \lambda_i \left(M_{Y_i}(r_i) - 1\right)\right)v}. \tag{5.3.20}$$

Thus, equation (5.3.20) yields the solution $k(v)$ given by relation (5.3.17) and the condition $k(0) = 1$ is equivalent to equation

$$h_\theta(r_1, ..., r_m) = M(r_1, ..., r_m) \int_0^\infty q(x) e^{-\left[\theta + \sum\limits_{i=1}^{m} \left(c_i r_i - \lambda_i M_{Y_i}(r_i) + \lambda_i\right)\right]x} dx - 1 = 0. \tag{5.3.21}$$

In order to complete the proof, we need to investigate the existence of solutions for equation (5.3.21). Note that

$$h_\theta(0, ..., 0) = \int_0^\infty q(x) e^{-\theta x} dx - 1 \leq \int_0^\infty q(x) dx - 1 = 0,$$

since $e^{-\theta x} \leq 1$ for $\theta \geq 0$. Also, we remark that for $\theta = 0$, then $h_0(0, ..., 0) = 0$.

Let k_2,...,k_m be non-negative real numbers and let us define

$$l_\theta(r_1) = h_\theta(r_1, k_2 r_1, ..., k_m r_1).$$

The first derivative of $l_\theta(r_1)$ is equal to

$$\frac{dl_\theta(r_1)}{dr_1} =$$

$$\left[\frac{\partial M(r_1, ..., r_m)}{\partial r_1} + \sum_{i=2}^{m} k_i \frac{\partial M(r_1, ..., r_m)}{\partial r_i}\right] \int_0^\infty q(x) e^{-\left[\theta + \sum\limits_{i=1}^{m} \left(c_i r_i - \lambda_i M_{Y_i}(r_i) + \lambda_i\right)\right]x} dx \Big|_{r_i = k_i r_1}$$

$$-M(r_1,...,k_m r_1)\left[c_1-\lambda_1\tfrac{dM_{Y_1}(r_1)}{dr_1}+\sum\limits_{i=2}^{m}\left(c_ik_i-\lambda_ik_i\tfrac{dM_{Y_i}(r_i)}{dr_1}\right)\right]$$
$$\times\int\limits_0^\infty xq(x)e^{-\left[\theta+\sum\limits_{i=1}^{m}\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}.$$

Therefore,

$$l'_\theta(0)=\left[\mu_1\int\limits_0^\infty q(x)e^{-\theta x}dx+\lambda_1\nu_1\int\limits_0^\infty xq(x)e^{-\theta x}dx-c_1\int\limits_0^\infty xq(x)e^{-\theta x}dx\right]$$
$$+\sum_{i=2}^{m}k_i\left[\mu_i\int\limits_0^\infty q(x)e^{-\theta x}dx+\lambda_i\nu_i\int\limits_0^\infty xq(x)e^{-\theta x}dx-c_i\int\limits_0^\infty xq(x)e^{-\theta x}dx\right].$$

For $\theta=0$, we obtain that $l'_0(0)<0$ since

$$\mu_i\int\limits_0^\infty q(x)dx+\lambda_i\nu_i\int\limits_0^\infty xq(x)dx-c_i\int\limits_0^\infty xq(x)dx=E[X_i]+\lambda_iE[Y_i]E[T]-c_iE[T]<0,$$

representing the net profit condition (5.2.5) for the i-th class, for $i=1,...,m$. Thus, it is proved that $l_0(r_1)$ is a decreasing function at zero.

In the case where $\theta>0$, we can not draw any conclusions regarding the sign of $l'_\theta(0)$.

The second derivative of $l_\theta(r_1)$ is equal to

$$\frac{d^2l_\theta(r_1)}{dr_1^2}=A(r_1,...,r_m)\int\limits_0^\infty q(x)e^{-\left[\theta+\sum\limits_{i=1}^{m}\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}$$
$$-2B(r_1,...,r_m)\left[c_1-\lambda_1\frac{dM_{Y_1}(r_1)}{dr_1}+\sum_{i=2}^{m}\left(c_ik_i-\lambda_ik_i\frac{dM_{Y_i}(r_i)}{dr_1}\right)\right]$$
$$\times\int\limits_0^\infty xq(x)e^{-\left[\theta+\sum\limits_{i=1}^{m}\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}$$
$$+M(r_1,...,r_m)\left[c_1-\lambda_1\frac{dM_{Y_1}(r_1)}{dr_1}+\sum_{i=2}^{m}\left(c_ik_i-\lambda_ik_i\frac{dM_{Y_i}(r_i)}{dr_1}\right)\right]^2$$
$$\times\int\limits_0^\infty x^2q(x)e^{-\left[\theta+\sum\limits_{i=1}^{m}\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}$$
$$+M(r_1,...,r_m)\left[\lambda_1\frac{d^2M_{Y_1}(r_1)}{dr_1^2}+\sum_{i=2}^{m}\lambda_ik_i^2\frac{d^2M_{Y_i}(r_i)}{dr_1^2}\right]$$

$$\times \int_0^\infty xq(x)e^{-\left[\theta+\sum\limits_{i=1}^m\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}, \tag{5.3.22}$$

where

$$A(r_1,...,r_m)=\frac{\partial^2 M(r_1,..,r_m)}{\partial r_1^2}+2\sum_{i=2}^m k_i\frac{\partial^2 M(r_1,..,r_m)}{\partial r_1\partial r_i}$$

$$+\sum_{i=2}^m k_i^2\frac{\partial^2 M(r_1,..,r_m)}{\partial r_i^2}+2\sum_{2\le i<j\le m}k_ik_j\frac{\partial^2 M(r_1,..,r_m)}{\partial r_i\partial r_j},$$

and

$$B(r_1,...,r_m)=\left[\frac{\partial M(r_1,...,r_m)}{\partial r_1}+\sum_{i=2}^m k_i\frac{\partial M(r_1,...,r_m)}{\partial r_i}\right].$$

Therefore, from (5.3.22) we further obtain

$$\frac{d^2 l_\theta(r_1)}{dr_1^2}=\int_0^\infty q(x)C(r_1,...,r_m)e^{-\left[\theta+\sum\limits_{i=1}^m\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}dx$$

$$+M(r_1,..,r_m)\left[\lambda_1E[Y_1^2e^{r_1Y_1}]+\sum_{i=2}^m\lambda_ik_i^2E[Y_i^2e^{r_iY_i}]\right]$$

$$\times\int_0^\infty xq(x)e^{-\left[\theta+\sum\limits_{i=1}^m\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx\Big|_{r_i=k_ir_1}>0,$$

where

$$C(r_1,...,r_m)=E\left[W^2e^{r_1X_1+...+r_mX_m}\right],\ \text{and}$$

$$W=X_1+\sum_{i=2}^m k_iX_i-x\left[c_1-\lambda_1E[Y_1e^{r_1Y_1}]+\sum_{i=2}^m k_i\left(c_i-\lambda_iE[Y_ie^{r_iY_i}]\right)\right].$$

This means that for every $\theta\ge 0$, the function $l_\theta(r_1)$ is convex in $r_1\in(0,\min\{r_1^0,r_1^1\})$. In the following, we summarize the properties obtained for the function h_θ.

For $\theta=0$, along every ray from the origin into $[0,\infty)^m$, $h_0(r_1,...,r_m)$ is a continuous, convex function, decreasing at zero and such that $h_0(0,...,0)=0$. These properties combined with the assumption that $\sup_{(r_1,...,r_m)\in R}h_0(r_1,...,r_m)>0$ enable us to conclude that the equation $h_0(r_1,...,r_m)=0$ admits at least one solution in R.

For $\theta>0$, along every ray from the origin into $[0,\infty)^m$, $h_\theta(r_1,r_2,...,r_m)$ is a continuous, convex function and $h_\theta(0,...,0)<0$. Therefore, since

$\sup_{(r_1,...,r_m)\in R} h_\theta(r_1,...,r_m) > 0$ it results that the equation $h_\theta(r_1,...,r_m) = 0$ admits at least one solution in R.

Consequently, for each $\theta \geq 0$, $(r_{1\theta},...,r_{m\theta}) \in R$ is determined by equation (5.3.18), and the vector process $\left\{k(V(t))e^{-\theta t}e^{-r_{1\theta}U_1(t)-...-r_{m\theta}U_m(t)}, t \geq 0\right\}$ is a martingale, with $k(v)$ defined by (5.3.17). $\square$

Following the proof of the above theorem, we note that for every $\theta \geq 0$ and for given non-negative constants, k_2,...,k_m, the equation

$$h_\theta(r_1, k_2r_1, ..., k_mr_1) = 0$$

has a unique solution in $(0, \min\{r_1^0, r_1^1\})$, provided that

$$\sup_{(r_1,k_2r_1,...,k_mr_1)\in R} h_\theta(r_1, k_2r_1, ..., k_mr_1) > 0.$$

In the following remark, we describe a situation where the assumption $\sup_{(r_1,...,r_m)\in R} h_\theta(r_1,...,r_m) > 0$ of Theorem 5.3.3 is satisfied.

Remark 5.3.1. *If $R = [0, r_1^1) \times ... \times [0, r_m^1)$, then the condition*

$$\sup_{(r_1,...,r_m)\in R} h_\theta(r_1,...,r_m) > 0,$$

imposed in Theorem 5.3.3*, is satisfied since the following property holds:*

$$\lim_{r_1\uparrow r_1^1,...,r_m\uparrow r_m^1} h_\theta(r_1,...,r_m) = \infty. \tag{5.3.23}$$

To show (5.3.23)*, we use the result regarding the classical risk model (presented in Section* 2.3*) namely,*

$$\lim_{r_i\uparrow r_i^1}[-c_ir_i + \lambda_i M_{Y_i}(r_i) - \lambda_i] = \infty.$$

Thus, for each $i = 1,...,m$ and for any $\epsilon > 0$ there exists a neighborhood, say $K_{i\epsilon}$, of r_i^1 ($0 < r_i^1 \leq \infty$), which depends on ϵ, such that

$$-\frac{\theta}{m} - c_ir_i + \lambda_i M_{Y_i}(r_i) - \lambda_i > \frac{\lambda}{m}(\epsilon+1) \tag{5.3.24}$$

holds for all $r_i \in K_{i\epsilon}$. Therefore, for all $(r_1,...,r_m) \in K_{1\epsilon} \times .. \times K_{m\epsilon}$,

$$M(r_1,...,r_m)\int_0^\infty q(x)e^{-\left[\theta+\sum\limits_{i=1}^m\left(c_ir_i-\lambda_iM_{Y_i}(r_i)+\lambda_i\right)\right]x}dx - 1 \geq \int_0^\infty q(x)e^{\lambda(\epsilon+1)x}dx - 1,$$

where we used inequality (5.3.24) *and the fact that $M(r_1,...,r_m) \geq 1$ for $r_i \geq 0$ ($i = 1,...,m$). Recall that $1/\lambda = E[T] = \int_0^\infty xq(x)dx$. Further, the well-known* *inequality* $e^x \geq x+1$ *for* $x \geq 0$ *yields that*

$$\int_0^\infty q(x)e^{\lambda(\epsilon+1)x}dx - 1 \geq \lambda(\epsilon+1)\int_0^\infty xq(x)dx + \int_0^\infty q(x)dx - 1 = \epsilon+1 > \epsilon. \quad (5.3.25)$$

By (5.3.25), *relation* (5.3.23) *follows.*

As an immediate consequence of Theorem 5.3.3, we have a similar result for the risk model given by (5.2.2).

Corollary 5.3.2. *Consider the risk model* (5.2.2) *and the function* $h_\theta^{(1)}$ *defined as*

$$h_\theta^{(1)}(r_1,...,r_m) = M(r_1,...,r_m)\int_0^\infty q(x)e^{-\left(\theta+\sum_{i=1}^m c_i r_i\right)x}dx - 1, \quad (5.3.26)$$

such that $\sup_{(r_1,...,r_m)\in R^{(1)}} h_\theta^{(1)}(r_1,...,r_m) > 0$, *where*

$$R^{(1)} = \{(r_1,..,r_m) \in [0,\infty)^m | M(r_1,...,r_m) < \infty\}. \quad (5.3.27)$$

Then for every $\theta \geq 0$, *the process*

$$Z_\theta^{(1)}(t) = k^{(1)}(V(t))e^{-\theta t}e^{-r_{1\theta}U_1(t)-...-r_{m\theta}U_m(t)}, \ t \geq 0,$$

is a martingale with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, *where*

$$k^{(1)}(v) = \frac{e^{\left(\theta+\sum_{i=1}^m c_i r_{i\theta}\right)v}M(r_{1\theta},.,r_{m\theta})}{1-Q(v)}\int_v^\infty q(x)e^{-\left(\theta+\sum_{i=1}^m c_i r_{i\theta}\right)x}dx,$$

and $(r_{1\theta},...,r_{m\theta}) \in R^{(1)}$ *is determined by the equation*

$$h_\theta^{(1)}(r_1,...,r_m) = 0.$$

We remark that in the univariate case ($m = 1$), the martingale $Z_\theta^{(1)}(t)$ established in Corollary 5.3.2 becomes the martingale obtained by Dassios and Embrechts (1989), which is illustrated by Proposition 2.3.20 (Subsection 2.3.4).

5.4. An upper bound for the infinite-time ruin probability of type ψ_{sim}

In this section, we use the martingales obtained in Theorem 5.3.3 and Corollary 5.3.2 in order to obtain upper bounds of Lundberg-type for the ruin probabilities of type $\psi_{sim}(u_1, ..., u_m)$.

Theorem 5.4.1. *Consider the risk model* (5.2.1) *and the function* h_0*, obtained by letting* $\theta = 0$ *in* (5.3.15)*, such that* $\sup_{(r_1,...,r_m)\in R} h_0(r_1, ..., r_m) > 0$. *Then*

$$\psi_{sim}(u_1, ..., u_m) \leq \inf_{(r_{10},...,r_{m0})\in S_0} \frac{1}{C(r_{10}, ..., r_{m0})} e^{-\sum_{i=1}^{m} r_{i0}u_i}, \tag{5.4.1}$$

where $S_0 = \{(r_{10}, ..., r_{m0}) \in R \mid h_0(r_{10}, ..., r_{m0}) = 0\}$ *and the constant* $C(r_{10}, ..., r_{m0})$ *(*$1 \leq C(r_{10}, ..., r_{m0}) < \infty$*) is defined by relation* (5.4.9).

Proof. Following the proof of Theorem 5.3.3, we have that the set S_0 is nonempty.

For a fixed $t > 0$, the stopping time $\tau_{sim} \wedge t = \min(\tau_{sim}, t)$ is bounded by t and the martingale stopping property, given by Proposition 2.1.4, can be applied for the martingale $Z_\theta(t)$ obtained in Theorem 5.3.3. Since for the ordinary renewal process $N(t)$ there is a claim at the origin, then $V(0) = 0$, and hence, we have

$$e^{-r_{1\theta}u_1-...-r_{m\theta}u_m} = E[Z_\theta(0)] = E[Z_\theta(t \wedge \tau_{sim})]$$

$$= E[Z_\theta(t), \tau_{sim} \leq t] + E[Z_\theta(\tau_{sim}), \tau_{sim} > t] \geq E[Z_\theta(\tau_{sim}), \tau_{sim} \leq t]$$
$$= E\left[k(V(\tau_{sim}))e^{-\theta\tau_{sim}} e^{-\sum_{i=1}^{m} r_{i\theta}U_i(\tau_{sim})} \mid \tau_{sim} \leq t\right] P\left(\tau_{sim} \leq t\right). \tag{5.4.2}$$

Since $U_i(\tau_{sim}) < 0$ on $(\tau_{sim} < \infty)$, for all $i = 1, 2, ..., m$, we have that

$$e^{-\sum_{i=1}^{m} r_{i\theta}U_i(\tau_{sim})} > 1.$$

Using this inequality in (5.4.2), we obtain

$$P(\tau_{sim} \leq t) \leq \frac{e^{-r_{1\theta}u_1-...-r_{m\theta}u_m}}{E\left[k(V(\tau_{sim}))\mid \tau_{sim} \leq t\right]} e^{\theta t}, \ \theta \geq 0. \tag{5.4.3}$$

Letting $t \to \infty$ in (5.4.3), under the restriction $\lim_{t\to\infty} e^{\theta t} < \infty$, yields

$$\psi_{sim}(u_1, ..., u_m) \leq e^{-r_{10}u_1-...-r_{m0}u_m} \lim_{t\to\infty} \frac{1}{E\left[k(V(\tau_{sim}))\mid \tau_{sim} \leq t\right]}, \tag{5.4.4}$$

where $(r_{10}, ..., r_{m0})$ is determined by the equation $h_0(r_1, ..., r_m) = 0$, and hence,

$$M(r_{10}, ..., r_{m0}) \int_0^\infty q(x) e^{-\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] x} dx = 1. \tag{5.4.5}$$

Now, using expression (5.3.17) of $k(v)$ yields for $\theta = 0$,

$$k(V(\tau_{sim})) = \frac{e^{\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] V(\tau_{sim})}}{1 - Q(V(\tau_{sim}))}$$

$$\times M(r_{10}, ..., r_{m0}) \int_{V(\tau_{sim})}^\infty q(x) e^{-\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] x} dx$$

$$\geq M(r_{10}, ..., r_{m0}) \int_{V(\tau_{sim})}^\infty q(x) e^{-\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] x} dx, \tag{5.4.6}$$

since $1 - Q(V(\tau_{sim})) < 1$ and $\sum_{i=1}^m (c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i) \geq 0$, which is a result of condition (5.4.5) combined with $M(r_{10}, ., r_{m0}) > 1$. Indeed, for $(r_{10}, ., r_{m0}) \in R$, we have that $M(r_{10}, ., r_{m0}) \geq 1$. On the other hand, if we suppose that $\sum_{i=1}^m (c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i) < 0$, then $\int_0^\infty q(x) e^{-\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] x} dx > 1$ and it would contradict relation (5.4.5).

If $\sigma_{N(\tau_{sim})} = \tau_{sim}$, then $V(\tau_{sim}) = 0$ and the right-hand side of inequality (5.4.6) becomes 1, in view of condition (5.4.5). Thus,

$$E\left[k(V(\tau_{sim}))\right| \tau_{sim} \leq t] \geq 1. \tag{5.4.7}$$

If $\sigma_{N(\tau_{sim})} < \tau_{sim}$, then $V(\tau_{sim}) = \tau_{sim} - \sigma_{N(\tau_{sim})} < T_{N(\tau_{sim})+1}$, and from inequality (5.4.6) we further obtain

$$k(V(\tau_{sim})) \geq M(r_{10}, ..., r_{m0}) \int_{T_{N(\tau_{sim})+1}}^\infty q(x) e^{-\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] x} dx.$$

Therefore, since $\{T_i\}_{i \geq 1}$ are independent and identically distributed, we obtain

$$E\left[k(V(\tau_{sim}))\right| \tau_{sim} \leq t] \geq M(r_{10}, ..., r_{m0}) E\left[\int_T^\infty q(x) e^{-\left[\sum_{i=1}^m \left(c_i r_{i0} - \lambda_i M_{Y_i}(r_{i0}) + \lambda_i\right)\right] x} dx\right]. \tag{5.4.8}$$

In view of (5.4.7) and (5.4.8), denoting

$$C(r_{10},.,r_{m0}) = \max\left\{ M(r_{10},...,r_{m0})E\left[\int\limits_T^\infty q(x)e^{-\left[\sum\limits_{i=1}^m\left(c_i r_{i0}-\lambda_i M_{Y_i}(r_{i0})+\lambda_i\right)\right]x}dx\right],1\right\}, \tag{5.4.9}$$

we obtain from inequality (5.4.4) that

$$\psi_{sim}(u_1,...,u_m) \leq \frac{1}{C(r_{10},...,r_{m0})}e^{-r_{10}u_1-...-r_{m0}u_m},$$

where $(r_{10},...,r_{m0})$ is an arbitrary solution of equation $h_0(r_1,...,r_m) = 0$. The proof is completed. □

In the following remark, we describe an upper bound for the ruin probability $\psi_1(u_1)$ associated to the univariate version of the risk model in (5.2.1).

Remark 5.4.1. *If we consider* $m = 1$ *for risk model* (5.2.1)*, then the upper bound of Theorem* 5.4.1 *becomes*

$$\psi_1(u_1) \leq \frac{1}{C(r_{10})}e^{-r_{10}u_1}, \tag{5.4.10}$$

where $r_{10} > 0$ *is such that* $M_{X_1}(r_{10}) < \infty$*,* $M_{Y_1}(r_{10}) < \infty$*, and satisfies* $h_0(r_{10}) = 0$*, which is obtained by taking* $\theta = 0$ *in* (5.3.15)*:*

$$M_{X_1}(r_{10})\int\limits_0^\infty q(x)e^{-\left[c_1 r_{10}-\lambda_1 M_{Y_1}(r_{10})+\lambda_1\right]x}dx = 1.$$

Also, the constant $C(r_{10})$*, defined by relation* (5.4.9)*, becomes in this case*

$$C(r_{10}) = \max\left\{M_{X_1}(r_{10})E\left[\int\limits_T^\infty q(x)e^{-\left[c_1 r_{10}-\lambda_1 M_{Y_1}(r_{10})+\lambda_1\right]x}dx\right],1\right\}.$$

In this particular case, the martingale established by Theorem 5.3.3*, which was used to derive the upper bound* (5.4.10)*, has the form*

$$Z_0(t) = k(V(t))e^{-r_{10}U_1(t)},\ t \geq 0,\ \textit{where} \tag{5.4.11}$$

$$k(v) = \frac{e^{\left[c_1 r_{10}-\lambda_1 M_{Y_1}(r_{10})+\lambda_1\right]v}M_{X_1}(r_{10})}{1-Q(v)}\int\limits_v^\infty q(x)e^{-\left[c_1 r_{10}-\lambda_1 M_{Y_1}(r_{10})+\lambda_1\right]x}dx.$$

Combining the upper bound (5.4.10) *with the upper bound obtained by Lv et al. (2010) given by Proposition* 5.2.6*, we can conclude that*

$$\psi_1(u_1) \leq \min\left\{\frac{1}{C(r_{10})}e^{-r_{10}u_1}, C(R)e^{-Ru_1}\right\},$$

where $C(R)$ *and* R *are defined in Proposition* 5.2.6*. As we discussed in Subsection* 5.2.1*, this upper bound of Lv et al. (2010) was obtained by using a martingale given by Proposition* 5.2.5 *and by considering the filtration* $\mathcal{F}_t = \mathcal{F}_t^{U_1} \vee \mathcal{F}_t^W$*, where* $W(t) = \sigma_{N(t)+1} - t$ *is the time remaining until the next renewal claim arrival. Note that even though the martingale described by Proposition* 5.2.5 *is simpler than the martingale given by property* (2) *of Proposition* 5.2.4 *obtained by Lv et al. (2010), the filtration* $\mathcal{F}_t^W$ *is not observable.*

We conclude this chapter by deriving an upper bound for the ruin probability of type $\psi_{sim}(u_1, ..., u_m)$ associated to the multivariate risk model (5.2.2).

Theorem 5.4.2. *Consider the risk model* (5.2.2) *and the function* $h_0^{(1)}$*, obtained by letting* $\theta = 0$ *in* (5.3.26)*, such that* $\sup_{(r_1,...,r_m)\in R^{(1)}} h_0^{(1)}(r_1, ..., r_m) > 0$*, where* $R^{(1)}$ *is defined by relation* (5.3.27)*. Then*

$$\psi_{sim}(u_1, ..., u_m) \leq \inf_{(r_{10},...,r_{m0})\in S_0^{(1)}} e^{-\sum_{i=1}^{m} r_{i0}u_i}, \tag{5.4.12}$$

where $S_0^{(1)} = \{(r_{10}, ..., r_{m0}) \in R^{(1)} \mid h_0^{(1)}(r_{10}, ..., r_{m0}) = 0\}$*.*

Proof. Following Corollary 5.3.2, we have that the set $S_0^{(1)}$ is non-empty. For a fixed $t > 0$, the stopping time $\tau_{sim} \wedge t = \min(\tau_{sim}, t)$ is bounded by t and the martingale stopping property, given by Proposition 2.1.4, can be applied for the martingale $Z_\theta^{(1)}(t)$ obtained in Corollary 5.3.2. For the ordinary renewal process $N(t)$ there is a claim at time zero, and thus, $V(0) = 0$. Furthermore, since $V(\tau_{sim}) = 0$ and hence, $k^{(1)}(V(\tau_{sim})) = k^{(1)}(0) = 1$, we have

$$e^{-r_{1\theta}u_1-...-r_{m\theta}u_m} = E[Z_\theta^{(1)}(0)] = E[Z_\theta^{(1)}(t \wedge \tau_{sim})]$$

$$= E[Z_\theta^{(1)}(t), \tau_{sim} \leq t] + E[Z_\theta^{(1)}(\tau_{sim}), \tau_{sim} > t] \geq E[Z_\theta^{(1)}(\tau_{sim}), \tau_{sim} \leq t]$$

$$= E\left[e^{-\theta\tau_{sim}}e^{-\sum_{i=1}^{m} r_{i\theta}U_i(\tau_{sim})} \mid \tau_{sim} \leq t\right] P\left(\tau_{sim} \leq t\right). \tag{5.4.13}$$

Since $U_i(\tau_{sim}) < 0$ on $(\tau_{sim} < \infty)$, for all $i = 1, 2, ..., m$, we have that

$$e^{-\sum\limits_{i=1}^{m} r_{i\theta} U_i(\tau_{sim})} > 1.$$

Using this inequality in (5.4.13), we obtain

$$P(\tau_{sim} \leq t) \leq e^{-r_{1\theta}u_1 - ... - r_{m\theta}u_m} e^{\theta t}, \ \theta \geq 0. \tag{5.4.14}$$

Letting $t \to \infty$ in (5.4.14), under the restriction $\lim_{t\to\infty} e^{\theta t} < \infty$, yields

$$\psi_{sim}(u_1, ..., u_m) \leq e^{-r_{10}u_1 - ... - r_{m0}u_m},$$

where $(r_{10}, ..., r_{m0})$ is determined by the equation $h_0^{(1)}(r_1, ..., r_m) = 0$, and therefore, it satisfies the following condition

$$M(r_{10}, ..., r_{m0}) \int_0^\infty q(x) e^{-\sum\limits_{i=1}^{m} c_i r_{i0} x} dx = 1.$$

Since $(r_{10}, ..., r_{m0})$ is an arbitrary solution of equation $h_0^{(1)}(r_1, ..., r_m) = 0$, the desired result is established. $\square$

5.5. Numerical illustrations

For numerical results purpose, we consider the bivariate model of type (5.2.2). This section contains numerical illustrations of $(r_{10}, k_2 r_{10})$-values with various non-negative k_2 and the corresponding values of the upper bounds of the form in (5.4.12) for the ruin probability $\psi_{sim}(u_1, u_2)$ with different choices of u_1 and u_2.

Thus, we need to solve the equation $h_0^{(1)}(r_{10}, k_2 r_{10}) = 0$, which is given by

$$h_0^{(1)}(r_{10}, k_2 r_{10}) = M_{X_1, X_2}(r_{10}, k_2 r_{10}) \int_0^\infty q(x) e^{-(c_1 r_{10} + c_2 k_2 r_{10})x} dx - 1 = 0. \tag{5.5.1}$$

We assume that the renewal process $N(t)$ is an Erlang(2) process; discussions on applications of the Erlang(n) processes in ruin theory can be found in Subsection 5.2.1. Thus, the renewal claim inter-arrival times $\{T_i\}_{i\geq 1}$ follow an Erlang (2, $\widetilde{\lambda}$) distribution and by Definition 1.1.1, the probability density function of T_i is $q(x) = \widetilde{\lambda}^2 x e^{-\widetilde{\lambda} x}$ for $x > 0$.

Let $\widetilde{\lambda} = 6$ and hence, the density of T_i is

$$q(x) = 36 x e^{-6x}, \quad x > 0, \tag{5.5.2}$$

and $E[T_i] = 2/\widetilde{\lambda} = 1/3$.

As in Section 4.8, we assume that the dependence structure between the light-tailed claim sizes X_1 and X_2 is defined by the bivariate Farlie-Gumbel-Morgenstern (FGM) copula given by Example 2.4.2 (Subsection 2.4.1). Let X_1 and X_2 be exponential random variables, that is, $F_1(x_1) = 1 - e^{-\alpha_1 x_1}$, for $\alpha_1 > 0$, $x_1 > 0$, and $F_2(x_2) = 1 - e^{-\alpha_2 x_2}$, for $\alpha_2 > 0$, $x_2 > 0$. Then the joint distribution function, $F(x_1, x_2)$, of (X_1, X_2) described by the bivariate FGM copula has the form

$$F(x_1, x_2) = (1 - e^{-\alpha_1 x_1})(1 - e^{-\alpha_2 x_2}) \left(1 + \rho e^{-\alpha_1 x_1} e^{-\alpha_2 x_2}\right),$$

while the joint density of (X_1, X_2) is

$$f(x_1, x_2) = \alpha_1 \alpha_2 e^{-\alpha_1 x_1} e^{-\alpha_2 x_2} \left[1 + \rho(1 - 2x_1)(1 - 2x_2)\right],$$

where $\rho \in [-1, 1]$ and the linear correlation coefficient is $\rho(X_1, X_2) = \rho/4$. Here, instead of using the notation θ for the dependence parameter as in Example 2.4.2, we use ρ in order to avoid any confusion regarding the parameter θ, which appears throughout this chapter.

In Section 4.8, formula (4.8.8) gives the moment generating function of (X_1, X_2) in this setting, that is,

$$M_{X_1,X_2}(r_1, r_2) = E[e^{r_1 X_1 + r_2 X_2}] = \alpha_1 \alpha_2 \frac{(2\alpha_1 - r_1)(2\alpha_2 - r_2) + \rho r_1 r_2}{(\alpha_1 - r_1)(2\alpha_1 - r_1)(\alpha_2 - r_2)(2\alpha_2 - r_2)}, \tag{5.5.3}$$

where $r_1 \in [0, \alpha_1)$ and $r_2 \in [0, \alpha_2)$.

We show how the upper bound established in Theorem 5.4.2 increases as the correlation coefficient ρ increases.

Using relation (5.5.3), we note that $M_{X_1,X_2}(r_1, r_2)$ is increasing in ρ and hence, $h_0^{(1)}(r_{10}, k_2 r_{10})$ from (5.5.1) is increasing in ρ, since $\int\limits_0^\infty q(x) e^{-(c_1 r_{10} + c_2 k_2 r_{10})x} dx > 0$.

Let $\rho^{(1)} \leq \rho^{(2)}$ and for given k_2 non-negative constant, $(r_{10}^{(l)}, k_2 r_{10}^{(l)})$ be solutions of the equation $h_0^{(1)}(r_{10}, r_{20})|_{\rho=\rho^{(l)}} = 0$, $l = 1, 2$. Then, since $h_0^{(1)}$ is increasing in ρ, we have that

$$\begin{aligned} h_0^{(1)}\left(r_{10}^{(1)}, k_2 r_{10}^{(1)}\right)|_{\rho=\rho^{(2)}} &\geq h_0^{(1)}\left(r_{10}^{(1)}, k_2 r_{10}^{(1)}\right)|_{\rho=\rho^{(1)}} = 0 \\ &= h_0^{(1)}\left(r_{10}^{(2)}, k_2 r_{10}^{(2)}\right)|_{\rho=\rho^{(2)}}. \end{aligned}$$

In the proof of Theorem 5.3.3, we established that $h_0^{(1)}(r_{10}, k_2 r_{10})$ is continuous, decreasing at zero, convex, and such that $h_0^{(1)}(0,0) = 0$, which lead to

$$h_0^{(1)}(r_{10}, k_2 r_{10})|_{\rho=\rho^{(2)}} < 0 \quad for \quad all \quad 0 < r_{10} < r_{10}^{(2)}, \tag{5.5.4}$$

$$\text{and} \quad h_0^{(1)}(r_{10}, k_2 r_{10})|_{\rho=\rho^{(2)}} > 0 \quad for \quad all \quad r_{10} > r_{10}^{(2)}. \tag{5.5.5}$$

Relations (5.5.4), (5.5.5) together with the result $h_0^{(1)}\left(r_{10}^{(1)}, k_2 r_{10}^{(1)}\right)|_{\rho=\rho^{(2)}} \geq 0$ yield $r_{10}^{(1)} \geq r_{10}^{(2)}$, and further,

$$e^{-r_{10}^{(1)}u_1 - k_2 r_{10}^{(1)}u_2} \leq e^{-r_{10}^{(2)}u_1 - k_2 r_{10}^{(2)}u_2}.$$

Therefore,

$$\inf e^{-r_{10}^{(1)}u_1 - k_2 r_{10}^{(1)}u_2} \leq \inf e^{-r_{10}^{(2)}u_1 - k_2 r_{10}^{(2)}u_2},$$

where the infimum is taken over $k_2 \in [0, \infty)$.

For numerical illustrations, let us assume that the mean claim sizes are $\mu_1 = 1/\alpha_1 = 5$, $\mu_2 = 1/\alpha_2 = 1$, and the safety loading coefficients are $\theta_1 = 0.5$, $\theta_2 = 0.4$, respectively. The premium rates, computed as $c_i = (1+\theta_i)/\left(E[T_i]\alpha_i\right)$, have the following values $c_1 = 22.5$ and $c_2 = 4.2$.

According to Proposition 2.2.6, we have that

$$m(t) = \sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty}\left[1 - \sum_{m=0}^{2n-1} \frac{e^{-6t}6^m t^m}{m!}\right] = \frac{6}{2}t - \frac{1}{4}(1 - e^{-12t}),$$

where $F_n(t)$, the distribution function of the n-th claim arrival time $\sigma_n = T_1 + ... + T_n$, is the distribution function of Erlang($2n$, 6) (by property (4) of Lemma 1.1.1).

Therefore, since $m(1) = 2.75$,

$$E[\sum_{k=1}^{N_1(1)} X_{1k}] = m(1)E[X_1] = 5m(1) = 13.75,$$

$$\text{and} \quad E[\sum_{k=1}^{N_2(1)} X_{2k}] = m(1)E[X_2] = m(1) = 2.75.$$

These results help us in setting the following values for the initial surpluses of the two classes: $u_1 = 13.75$ and $u_2 = 2.75$. In order to see the impact of the initial surpluses on the upper bounds for the ruin probability $\psi_{sim}(u_1, u_2)$, we also consider the following values of (u_1, u_2): $(13.75, 7)$, $(30, 2.75)$ and $(30, 7)$.

Using relations (5.5.2), (5.5.3) and all numerical values, we proceed to solve equation (5.5.1), by considering the following values for k_2: 0.01, 1, and 10 and for the coefficient ρ: -0.3, -0.7, 0, 0.3 and 0.7. The values obtained for r_{10} and

TABLE 5.1. Values of r_{10} and $e^{-r_{10}u_1-0.01r_{10}u_2}$ for $k_2 = 0.01$.

ρ	$u_1 = 13.75$	$u_1 = 13.75$	$u_1 = 30$	$u_1 = 30$
	$u_2 = 2.75$	$u_2 = 7$	$u_2 = 2.75$	$u_2 = 7$
-0.7	0.0850384	0.0850384	0.0850384	0.0850384
	0.3098656	0.3087477	0.0778095	0.0775288
-0.3	0.0850231	0.0850231	0.0850231	0.0850231
	0.3099309	0.3088131	0.0778453	0.0775645
0	0.0850117	0.0850117	0.0850117	0.0850117
	0.3099796	0.3088617	0.0778720	0.0775911
0.3	0.085003	0.085003	0.085003	0.085003
	0.3100167	0.3088988	0.0778923	0.0776114
0.7	0.0849851	0.0849851	0.0849851	0.0849851
	0.3100932	0.3089753	0.0779342	0.0776532

TABLE 5.2. Values of r_{10} and $e^{-r_{10}u_1-r_{10}u_2}$ for $k_2 = 1$.

ρ	$u_1 = 13.75$	$u_1 = 13.75$	$u_1 = 30$	$u_1 = 30$
	$u_2 = 2.75$	$u_2 = 7$	$u_2 = 2.75$	$u_2 = 7$
-0.7	0.0874354	0.0874354	0.0874354	0.0874354
	0.2362926	0.1629544	0.0570681	0.0393558
-0.3	0.0859192	0.0859192	0.0859192	0.0859192
	0.2422786	0.1681626	0.0599734	0.0416268
0	0.0848232	0.0848232	0.0848232	0.0848232
	0.2466998	0.1720308	0.0621652	0.0433495
0.3	0.0837604	0837604	0837604	0837604
	0.2510641	0.1758668	0.0643670	0.0450881
0.7	0.0823919	0.0823919	0.0823919	0.0823919
	0.2567977	0.1809323	0.0673175	0.0474300

for the upper bound $e^{-r_{10}u_1-k_2r_{10}u_2}$, assuming the three aforementioned values of k_2, are shown in Tables 5.1, 5.2 and 5.3, respectively.

TABLE 5.3. Values of r_{10} and $e^{-r_{10}u_1-10r_{10}u_2}$ for $k_2 = 10$.

ρ	$u_1 = 13.75$	$u_1 = 13.75$	$u_1 = 30$	$u_1 = 30$
	$u_2 = 2.75$	$u_2 = 7$	$u_2 = 2.75$	$u_2 = 7$
-0.7	0.0401507	0.0401507	0.0401507	0.0401507
	0.1908597	0.0346443	0.0993938	0.0180416
-0.3	0.0385543	0.0385543	0.0385543	0.0385543
	0.2038512	0.0396001	0.1089493	0.0211645
0	0.0374649	0.0374649	0.0374649	0.0374649
	0.2132207	0.0433830	0.1159922	0.0236004
0.3	0.0364539	0.0364539	0.0364539	0.0364539
	0.2223009	0.0472163	0.1229350	0.0261112
0.7	0.0352108	0.0352108	0.0352108	0.0352108
	0.2339973	0.0523970	0.1320438	0.0295674

We remark that for each choice of k_2, illustrated by each of the Tables 5.1, 5.2 and 5.3, the values of r_{10} and hence, of $k_2 r_{10}$ decrease as the coefficient ρ increases, which implies that the corresponding values of the upper bound $e^{-r_{10}u_1-k_2r_{10}u_2}$ for ruin probability $\psi_{sim}(u_1, u_2)$ increase. This confirms the theoretical property established earlier.

In the case illustrated by Table 5.1 where $k_2 = 0.01$, for each choice of ρ, the upper bound decreases as initial surpluses increase from $(13.75, 2.75)$ to $(30, 7)$. Same remark is valid for the values in Table 5.2, where $k_2 = 1$. Regarding Table 5.3 where $k_2 = 10$, the upper bound increases as we go from $(u_1 = 13.75, u_2 = 7)$ to $(u_1 = 30, u_2 = 2.75)$, but the upper bound is smallest for the values $(u_1 = 30, u_2 = 7)$.

Comparing Tables 5.1 to 5.3, we note that for given ρ and (u_1, u_2), the upper bound is most sensitive to k_2 and not to u_1, u_2 or ρ.

5.6. CONCLUSIONS

In this chapter, we set up a multivariate common shock framework for losses of different types that allows for both dependence in claim frequencies across types and dependence in claim sizes. An interpretation of the model studied in this chapter is that for each class of insurance business the customary claim occurrences are modeled by a Poisson process which affects only one class, while the others are unaffected, and a renewal process counts the number of common shocks that affect all classes of business and brings about claims represented by possibly dependent random variables. The common shock can be viewed for example, as an earthquake, a tsunami, or a tornado.

For this multivariate risk model, we employed the backward Markovization technique in order to identify a Markov vector process and using tools from the theory of piecewise deterministic Markov processes leads to constructing an exponential martingale needed in our ruin problem. More specifically, based on this martingale, we derived Lundberg-type upper bounds for the probability that ruin occurs in all classes simultaneously, for the multivariate risk model aforementioned, and also for a special case where the individual shocks are absent and the claims across classes are produced only by the renewal process. For the latter, we provided numerical results for these upper bounds, where a bivariate version is considered and the dependence in claim sizes is created using copula techniques.

Following the numerical results, we conclude that the upper bound for the ruin probability $\psi_{sim}(u_1, u_2)$ is increasing in the correlation parameter ρ, which reflects the dependence between claim sizes.

Chapter 6

CONCLUSIONS AND FUTURE WORK

In this dissertation, we have proposed and investigated models of dependence with potential applications in actuarial science.

We proposed a new class of bivariate Erlang distributions, which can be viewed as fatal shock models or competing risk models useful in various fields such as the actuarial theory of life insurance, finance and reliability theory.

We have established properties of the bivariate Erlang distribution such as representation of the joint survival function as a mixture of an absolutely continuous part and a singular part, joint density function, marginal distributions, conditional densities, conditional expectations, Laplace transform, moments, correlation and covariance structure. Also, a finite mixture of bivariate Erlang distributions is obtained.

Unfortunately, statistical inference for the BVEr distribution is not an easy task because of the complicated nature of its density function. Due to the constraints on the shape parameters of the Erlang distributions to be positive integers, the problem of estimating all six parameters which define the bivariate Erlang distribution is challenging. Consequently, assuming the shape parameters known, we have been successful in estimating the other three parameters using an Expectation-Maximization algorithm and simulations have been carried out to see the performance of the estimator.

We have also worked on extending this Expectation-Maximization algorithm to the situation where the shape parameters are unknown and obtained promising results. However, there is a need for further investigation to formulate an estimation method regarding all parameters. This study will be followed by considering an application of the bivariate Erlang distribution in life insurance modeling. This work is in progress.

In regard to risk theory, we investigated multivariate risk processes which may be useful in studying ruin problems for insurance companies handling dependent classes of business.

We first considered a multivariate risk process, modeling the dependence through the number of claims using the Poisson model with common shocks. The latter assumes that in addition to the individual shocks, a common shock affects all classes of business and that another common shock has an impact on each couple of classes. Further, it was assumed that this multivariate risk process is perturbed by diffusion which is characterized by a multivariate correlated Brownian motion.

In a more general setting, we also studied a multivariate renewal process, which assumes that for each class of insurance business the customary claim occurrences are modeled by a Poisson process which affects only one class, while the others are unaffected, and a renewal process counts the number of common shocks that affect all classes of business. Dependence between claims sizes across classes is allowed.

For all of these multivariate risk processes, we have applied results from the theory of piecewise deterministic Markov processes in order to derive exponential martingales. The latter are needed to establish computable upper bounds for the ruin probabilities whose expressions are intractable. In this sense, Lundberg-type upper bounds for the probability that ruin occurs in all classes simultaneously have been obtained for these models. Also, the probability that ruin occurs in at least one class of business and the asymptotic behavior of the probability that ruin occurs in all classes simultaneously before time t have been studied.

Numerical results regarding the upper bounds for the probability that ruin occurs in all classes simultaneously are reported for particular cases of these models such as bivariate and trivariate risk processes, where the dependence in claim sizes is created using copula techniques. It has been shown that the upper bound of the ruin probability is increasing by adding a diffusion process and also, by increasing the dependence between claim sizes.

Our discussion of the multivariate model based on a common renewal shock can offer a beneficial context for further research. For example, we think of considering a version of this multivariate model, where the common shock that affects all classes of insurance business is a compound non-homogeneous Poisson process with periodic claim intensity rate. This model would be of interest for insurance portfolios under seasonal environments.

It would also be nice to investigate ruin probabilities in the case where the dependence between claim amounts for a portfolio of two classes of insurance business is modeled by the bivariate Erlang distribution proposed in this thesis.

Note that for the risk processes studied in this thesis, it is assumed that the claim inter-arrival times between two successive claims and the claim amounts are independent. However, this assumption does not hold in reality. For example, larger damages caused by earthquakes or tsunami are expected with a longer period between claims. Therefore, it would be interesting to assume for the multivariate renewal model investigated in this thesis that the claim sizes and the renewal claim inter-arrival times are dependent. This project is a subject of future studies.

BIBLIOGRAPHY

[1] A. Al-Khedhairi and A. El-Gohary, *A new class of bivariate Gompertz distributions and its mixture*, International Journal of Mathematical Analysis, **5**, No. 2, 235-253, 2008.

[2] R.S. Ambagaspitiya, *On the distribution of a sum of correlated aggregate claims*, Insurance: Mathematics and Economics, **23**, 15-19, 1998.

[3] R.S. Ambagaspitiya, *On the distributions of two classes of correlated aggregate claims*, Insurance: Mathematics and Economics, **24**, 301-308, 1999.

[4] R.S. Ambagaspitiya, *Aggregate survival probability of a portfolio with dependence*, Insurance: Mathematics and Economics, **32**, 431-443, 2003.

[5] E. S. Andersen, *On the collective theory of risk in case of contagion between claims*, In: Transactions XVth International Congress of Actuaries, New York, II, 219-229, 1957.

[6] S. Asmussen, *Applied probability and queues*, Wiley, New York, 1987.

[7] S. Asmussen, *Risk theory in a Markovian environment*, Scandinavian Actuarial Journal, 69-100, 1989.

[8] S. Asmussen, *Ruin probabilities*, Advanced Series on Statistical Science and Applied Probability, Vol. 2, World Scientific, Singapore, 2000.

[9] S. Asmussen and H. Albrecher, *Ruin probabilities*, Second Ed. World Scientific, New Jersey, 2010.

[10] R.E. Barlow and F. Proschan, *Statistical theory of reliability and life testing. Probability models*, Holt, Rinehart and Winston, New York, 1975.

[11] A.P. Basu and H.W. Block, *On characterizing univariate and multivariate exponential distributions with applications*, Statistical Distributions in Scientific Work, Vol.3, In: Patil, G.P., Kotz, S., Ord, J., eds., Dordrecht: D.Reidel, 399-421, 1975.

[12] R.E. Beard, T. Pentikäinen and M. Pesonen, *Risk theory*, 3rd ed., Chapman and Hall, London, 1984.

[13] J. Beekman, *Collective risk results*, Transactions of Society of Actuaries, **20**, 182-199, 1968.

[14] G.K. Bhattacharyya and R.A. Johnson, *Maximum likelihood estimation and hypothesis testing in the bivariate exponential model of Marshall and Olkin*, Technical report, No. 276, Department of Statistics, University of Wisconsin, 1971.

[15] N.H. Bingham, C.M. Goldie and J.L. Teugels , *Regular variation*, Cambridge University Press, 1987.

[16] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones and C.J. Nesbitt, *Actuarial Mathematics*, Second Ed., Schaumburg, Ill : Society of Actuaries, 1997.

[17] H. Bühlmann, *Mathematical methods in risk theory*, Springer, New York, U.S.A., 1970.

[18] J. Cai and J. Garrido, *A unified approach to the study of tail probabilities of compound distributions*, Journal of Applied Probability, **36**, 1058-1073, 1999a.

[19] J. Cai and J. Garrido, *Two-sided bounds for ruin probabilities when the adjustment coefficient does not exist*, Scandinavian Actuarial Journal, 80-92, 1999b.

[20] J. Cai and H. Li, *Multivariate risk model of phase type*, Insurance: Mathematics and Economics, **36**, 137-152, 2005.

[21] J. Cai J. and H. Li, *Dependence properties and bounds for ruin probabilities in multivariate compound risk models*, Journal of Multivariate Analysis, **98**, 757-773, 2007.

[22] W. Chan, H. Yang and L. Zhang, *Some results on ruin probabilities in a two-dimensional risk model*, Insurance: Mathematics and Economics, **32**, 345-358, 2003.

[23] Y. Chen, K.C. Yuen and K.W. Ng, *Asymptotics for the ruin probabilities of a two-dimensional renewal risk model with heavy-tailed claims*, Journal of Applied Stochastic Models in business and industry, Vol. 27, **3**, 290-300, 2011.

[24] E. Çinlar, *Introduction to stochastic processes*, Prentice-Hall, Englewood Cliffs, NJ, U.S.A., 1975.

[25] H. Cossette and É. Marceau, *The discrete-time risk model with correlated classes of business*, Insurance: Mathematics and Economics, **26**, 133-149, 2000.

[26] H. Cossette, É Marceau and F. Marri, *On the compuond Poisson risk model with dependence based on a generalized FGM copula*, Insurance: Mathematics and Economics, **43**, 444-455, 2008.

[27] H. Cossette, É Marceau and F. Marri, *Analysis of ruin measures for the classical compound Poisson risk model with dependence*, Scandinavian Actuarial Journal, **3**, 221-245, 2010.

[28] D.R. Cox, *The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables*, Proc. Camb. Philos. Soc., **51**, 433-441, 1955.

[29] D.R. Cox, *The analysis of exponentially distributed lifetimes with two types of failure*, Journal of the Royal Statistical Society, Series B, **21**, 411-421, 1959.

[30] D.R. Cox, *Renewal theory*, Butler and Tanner, 1967.

[31] H. Cramér, *On the Mathematical Theory of Risk*, Skandia Jubilee Volume, Stockholm, Sweden, 1930.

[32] H. Cramér, *Collective Risk Theory*, Skandia Jubilee Volume, Stockholm, Sweden, 1955.

[33] L. Dang, N. Zhu and H. Zhang, *Survival probability for a two-dimensional risk model*, Insurance: Mathematics and Economics, **44**, 491-496, 2009.

[34] A. Dassios and P. Embrechts, *Martingales and insurance risk*, Communications in Statistics-Stochastic Models, **5**, No.2, 181-217, 1989.

[35] A. Dassios and J. Jang, *Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity*, Finance and Stochastics, **7**, No.1, 73-95, 2003.

[36] M.H.A. Davis, *Piecewise deterministic Markov processes: A general class of non-diffusion stochastic models*, Journal of the Royal Statistical Society, Series B, Vol.46, **3**, 353-388, 1984.

[37] M.H.A. Davis, *Markov models and optimization*, Chapman and Hall, London, 1993.

[38] A.P. Dempster, N.M. Laird and D.B. Rubin, *Maximum likelihood estimation from incomplete data via the EM algorithm*, Journal of the Royal Statistical Society, Series B (Methodological), **39**, No.1, 1-38, 1977.

[39] F. DeVylder, *Martingales and ruin in a dynamic risk process*, Scandinavian Actuarial Journal, **217**, 1977.

[40] F. DeVylder and M. Goovaerts, *Bounds for classical ruin probabilities*, Insurance: Mathematics and Economics, **3**, 121-131, 1984.

[41] H.F. Diaz and R.J. Murnane, *Climate extremes and society*, Cambridge University Press, New York, 2008.

[42] D. Dickson, *An upper bound for the probability of ultimate ruin*, Scandinavian Actuarial Journal, 131-138, 1994.

[43] D. Dickson, *On a class of renewal risk processes*, North American Actuarial Journal, **2**, No.3, 60-68, 1998.

[44] D. Dickson, *Insurance risk and ruin*, Cambridge University Press, Cambridge, 2005.

[45] D. Dickson and C. Hipp, *Ruin probabilities for Erlang(2) risk processes*, Insurance: Mathematics and Economics, **22**, 251-262, 1998.

[46] D. Dickson and C. Hipp, *On the time to ruin for Erlang(2) risk processes*, Insurance: Mathematics and Economics, **29**, 333-344, 2001.

[47] F. Dufresne and H.U. Gerber , *Risk theory for the compound Poisson process that is perturbed by diffusion*, Insurance: Mathematics and Economics, **10**, No. 1, 51-59, 1991.

[48] R. Durrett, *Essentials of stochastic processes*, Springer-Verlag, New York, 1999.

[49] E.B. Dynkin, *Markov processes*, I, Springer Verlag, Berlin, 1965.

[50] P. Embrechts and N. Veraverbeke, *Estimates for the probability of ruin with special emphasis on the possibility of large claims*, Insurance: Mathematics and Economics, **1**, No. 1, 55-72, 1982.

[51] P. Embrechts, C. Klüppelberg and T. Mikosch, *Modeling extremal events for insurance and finance*, Springer-Verlag, Berlin, 1997.

[52] A.K. Erlang, *Solution of some problems in the theory of probabilities of significance in automatic telephone exchange*, Post Off. Electr. Eng. Journal, **10**, 189-197, 1917.

[53] J.D. Esary, A.W. Marshall and F. Proschan, *Shock models and wear processes*, The Annals of Probability, **1**, No. 4, 627-649, 1973.

[54] W. Feller, *An introduction to probability theory and its applications*, Second edition, Volume 2, John Wiley and Sons, New York, 1971.

[55] E. Frees, *Data analysis using regression models: the business perspective*, Englewood Cliffs: Prentice Hall, 1996.

[56] E. W. Frees, J. Carriere and E. Valdez, *Annuity valuation with dependent mortality*, The Journal of Risk and Insurance, **63**, No. 2, 229-261, 1996.

[57] E. W. Frees and E. Valdez, *Understanding relationships using copulas*, North American Actuarial Journal, **2**, No. 1, 1-25, 1998.

[58] H.J. Furrer and H. Schmidli, *Exponential inequalities for ruin probabilities of risk processes perturbed by diffusion*, Insurance: Mathematics and Economics, **15**, 23-36, 1994.

[59] J. Galambos and S. Kotz, *Characterizations of probability distributions*, Berlin: Springer, 1978.

[60] H.U. Gerber, *An extension of the renewal equation and its application in the collective theory of risk*, Scandinavian Actuarial Journal, **53**, 205-210, 1970.

[61] H.U. Gerber, *Martingales in risk theory*, Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker, **73**, 205-216, 1973.

[62] H.U. Gerber, *An introduction to mathematical risk theory*, S.S. Heubner Foundation Monograph Series 8, University of Pennsylvania, Philadelphia, 1979.

[63] H. Gerber and E.S.W. Shiu, *The time value of ruin in a Sparre Andersen model*, North American Actuarial Journal, **9**, No. 2, 49-69, 2005.

[64] K. Giescke, *A simple exponential model for dependent defaults*, Journal of Fixed Income, **13**, No. 3, 74-83, 2003.

[65] L. Gong, A. Badescu and E. Cheung, *Recursive methods for a multi-dimensional risk process with common shocks*, Insurance: Mathematics and Economics, **50**, 109-120, 2012.

[66] M.J. Goovaerts, R. Kaas, A.E. Van Heerwaarden and T. Bauwelinckx, *Effective Actuarial Methods*, North Holland, Amsterdam, 1990.

[67] J. Grandell, *Aspects of risk theory*, Springer-Verlag, New York, 1991.

[68] P. Halmos, *Measure theory*, 2nd edition, Springer, New York, 1978.

[69] C. Jagger and C.J. Sutton, *Death after marital bereavement-is the risk increased?*, Statistics in Medicine, **10**, 395-404, 1991.

[70] J. Jang, *Martingale approach for moments of discounted aggregate claims*, The Journal of Risk and Insurance, **71**, No.2, 201-211, 2004.

[71] J. Jang, *Jump diffusion processes and their applications in insurance and finance*, Insurance: Mathematics and Economics, **41**, No.1, 62-70, 2007.

[72] H. Joe, *Multivariate models and dependence concepts*, Chapman and Hall, London, 1997.

[73] R. Kaas and Q. Tang, *Note on the tail behavior of random walk maxima with heavy tails and negative drift*, North American Actuarial Journal, Vol. 7, **3**, 57-61, 2003.

[74] D. Kannan, *An introduction to stochastic processes*, Elsevier North Holland, Inc., New York, 1979.

[75] D. Karlis, *ML estimation for multivariate shock models via an EM algorithm*, Annals of the Institute of Statistical Mathematics, **55**, 817-830, 2003.

[76] S. Klugman, H. Panjer and G.E. Willmot, *Loss Models: From data to decisions*, Second edition, Wiley and Sons, Inc., New Jersey, 2004.

[77] B. Ko and Q. Tang, *Sums of dependent nonnegative random variables with subexponential tails*, Journal of Applied Probability, **45**, 85-94, 2008.

[78] J. Komelj and M. Perman, *Joint characteristic functions construction via copulas*, Insurance: Mathematics and Insurance, **47**, 137-143, 2010.

[79] S. Kotz, N. Balakrishman and N.L. Johnson, *Continuous multivariate distributions*, John Wiley and Sons, Inc., New York, 2000.

[80] D. Kundu and A.K. Dey, *Estimating the parameters of the Marshall-Olkin bivariate Weibull distribution by EM algorithm*, Computational Statistics and Data Analysis, **53**, No. 4, 956-965, 2009.

[81] D. Kundu and R.D. Gupta, *Modified Sarhan-Balakrishnan singular bivariate distribution*, Journal of Statistical Planning and Inference, **140**, 526-538, 2010.

[82] C.K. Lee and X.S. Lin, *Modeling dependent risks with multivariate Erlang mixtures*, Astin Bulletin, **42**, No. 1, 153-180, 2012.

[83] R. Leipus and J. Siaulys, *Asymptotic behavior of the finite time ruin probability under subexponential claim sizes*, Insurance: Mathematics and Economics, **40**, No. 3, 498-508, 2007.

[84] J. Li, Z. Liu and Q. Tang, *On the ruin probabilities of a bidimensional perturbed risk model*, Insurance: Mathematics and Economics, **41**, 185-195, 2007.

[85] S. Li, *A note on the maximum severity of ruin in an Erlang(n) risk process*, Bulletin Swiss Association of Actuaries, 167-180, 2008.

[86] S. Li and D. Dickson, *The maximum surplus before ruin in an Erlang(n) risk process and related problems*, Insurance: Mathematics and Economics, **38**, 529-539, 2006.

[87] S. Li and J. Garrido, *On ruin for the Erlang(n) risk process*, Insurance: Mathematics and Economics, **34**, 391-408, 2004.

[88] S. Li and J. Garrido, *Ruin probabilities for two classes of risk processes*, Astin Bulletin, Vol. 35, **1**, 61-77, 2005.

[89] F. Lindskog and A.J. McNeil, *Common Poisson shock models: applications to insurance and credit risk modelling*, Astin Bulletin, **33**, No.2, 209-238, 2003.

[90] G.X. Liu, Y. Wang and B. Zhang, *Ruin probabilities in the continuous-time compound binomial model*, Insurance: Mathematics and Economics, **36**, 303-316, 2005.

[91] G. Lu, L. Guoxin and H. Liyan, *Some results about ruin theory in the continuous-time*, Proceedings of 2007 IEEE International Conference on Grey Systems and Intelligent Services, China, 1533-1537, 2007.

[92] F. Lundberg, *Försäkringsteknisk Risktujämning*, F. Englunds, A.B. Boktryckeri, Stockholm, 1926.

[93] F. Lundberg, *Some suplementary researches on the collective risk theory*, Skandinavisk Aktuarietidskrift, **15**, 137-158, 1932.

[94] T.L. Lv, J.Y. Guo and X. Zhang, *Ruin probabilities for a risk model with two classes of claims*, Acta Mathematica Sinica, English Series, **26**, No. 9, 1-12, 2010.

[95] K.V. Mardia, *Families of bivariate distributions*, Griffin's Statistical Monograph, **27**, Charles Griffin, London, 1970.

[96] A.W. Marshall and I. Olkin, *A multivariate exponential distribution*, Journal of the American Statistical Association, **62**, 30-44, 1967a.

[97] A.W. Marshall and I. Olkin, *A generalized bivariate exponential distribution*, Journal of Applied Probability, **4**, 291-302, 1967b.

[98] A.W. Marshall and I. Olkin, *Families of multivariate distributions*, Journal of the American Statistical Association, **83**, 834-841, 1988.

[99] G. McLachlan and N. Krishnan, *The EM algorithm and extensions*, Wiley, Chichester, 1997.

[100] W. Mendenhall and R.J. Hader, *Estimation of parameters of mixed exponentially distributed failure time distributions from censored life test data*, Biometrika, **45**, 504-520, 1958.

[101] R.B. Nelsen, *An introduction to copulas*, 2nd ed., Springer Science and Business Media, Inc., New York, 2006.

[102] R. Nelson, *Probability, Stochastic processes, and Queueing theory*, Springer, 1995.

[103] H.H. Panjer and G.E. Willmot, *Insurance risk models*, The Society of Actuaries, 1992.

[104] C.M. Parkes, B. Benjamin and R.G. Fitzgerald, *Broken heart: a statistical study of increased mortality among widowers*, British Medical Journal, **1**, 740-743, 1969.

[105] T. Rolski, H. Schmidli, V. Schmidt and J.L. Teugels, *Stochastic Processes for Insurance and Finance*, Wiley, New York, 1999.

[106] S.M. Ross, *Stochastic processes*, 2nd Edition, Wiley, New York, 1996.

[107] A.M. Sarhan and N. Balakrishnan, *A new class of bivariate distributions and its mixture*, Journal of Multivariate Analysis, **98**, 1508-1527, 2007.

[108] S. Schlegel, *Ruin probabilities in perturbed risk models*, Insurance: Mathematics and Economics, **22**, 93-104, 1998.

[109] H. Schmidli, *Perturbed risk processes: A review*, Theory of stochastic processes, **5**, 145-165, 1999.

[110] H. Schmidli, *Conditional law of risk processes given that ruin occurs*, Insurance: Mathematics and Economics, **46**, 281-289, 2010.

[111] H. L. Seal, *The Poisson process: its failure in risk theory*, Insurance: Mathematics and Economics, **2**, No. 4, 287-288, 1983.

[112] A. Sklar, *Fonctions de répartition à n dimensions et leurs marges*, Publications de l'Institut Statistique de l'Université de Paris, **8**, 229-231, 1959.

[113] L. Takács, *Introduction to the theory of queues*, Oxford University Press, New York, 1962.

[114] Q. Tang, *Asymptotics for the finite time ruin probability in the renewal model with consistent variation*, Stochastic models, **20**, No. 3, 281-297, 2004.

[115] Q. Tang and G. Tsitsiashvili, *Precise estimates for the ruin probability in infinite horizon in a discrete-time model with heavy tailed insurance and financial risks*, Stochastic processes and their applications, **108**, No. 2, 299-325, 2003.

[116] J.L.Teugels and N. Veraverbeke N., *Cramér-type estimates for the probability of ruin*, C.O.R.E. Discussion paper No. 7316, 1973.

[117] H. Tijms, *Stochastic Models: An algorithm approach*, Chicester: John Wiley, 1994.

[118] O. Thorin, *On the asymptotic behavior of the ruin probability for an infinite period when the epochs of claims form a renewal process*, Scandinavian Actuarial Journal, **57**, 81-99, 1974.

[119] P. Veenus and K.R.M. Nair, *Characterization of a bivariate Pareto distribution*, Journal of Indian Statistical Association, **32**, 15-20, 1994.

[120] N. Veraverbeke, *Asymptotic estimates for the probability of ruin in a Poisson model with diffusion*, Insurance: Mathematics and Economics, **13**, 57-62, 1993.

[121] S. Wang, *Aggregation of correlated risk portfolios: models and algorithms*, Proceedings of the Casualty Actuarial Society, **85**, 848-939, 1998.

[122] G. Wang and K. Yuen, *On a correlated aggregate claims model with thinning-dependence structure*, Insurance: Mathematics and Economics, **36**, 456-468, 2005.

[123] G.E. Willmot and X.S. Lin, *Lundberg approximations for compound distributions with insurance applications*, Springer-Verlag, New York, 2001.

[124] W.H. Young, *On multiple integration by parts and the second theorem of the mean*, Proceedings London Mathematical Society, Series 2, **16**, 273-293, 1917.

[125] K. Yuen, J. Guo and X. Wu, *On a correlated aggregate claims model with Poisson and Erlang risk processes*, Insurance: Mathematics and Economics, **31**, 205-214, 2002.

[126] K. Yuen and G. Wang, *Comparing two models with dependent classes of business*, Actuarial research clearing house, Society of Actuaries, Volume 1, p. 22, 2002.

[127] K. Yuen, J. Guo and X. Wu, *On the first time of ruin in the bivariate compound Poisson model*, Insurance: Mathematics and Economics, **38**, 298-308, 2006.